Decoding the Past: Rejecting a Dynamic Hopf Model.

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February 18, 2014

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Introduction

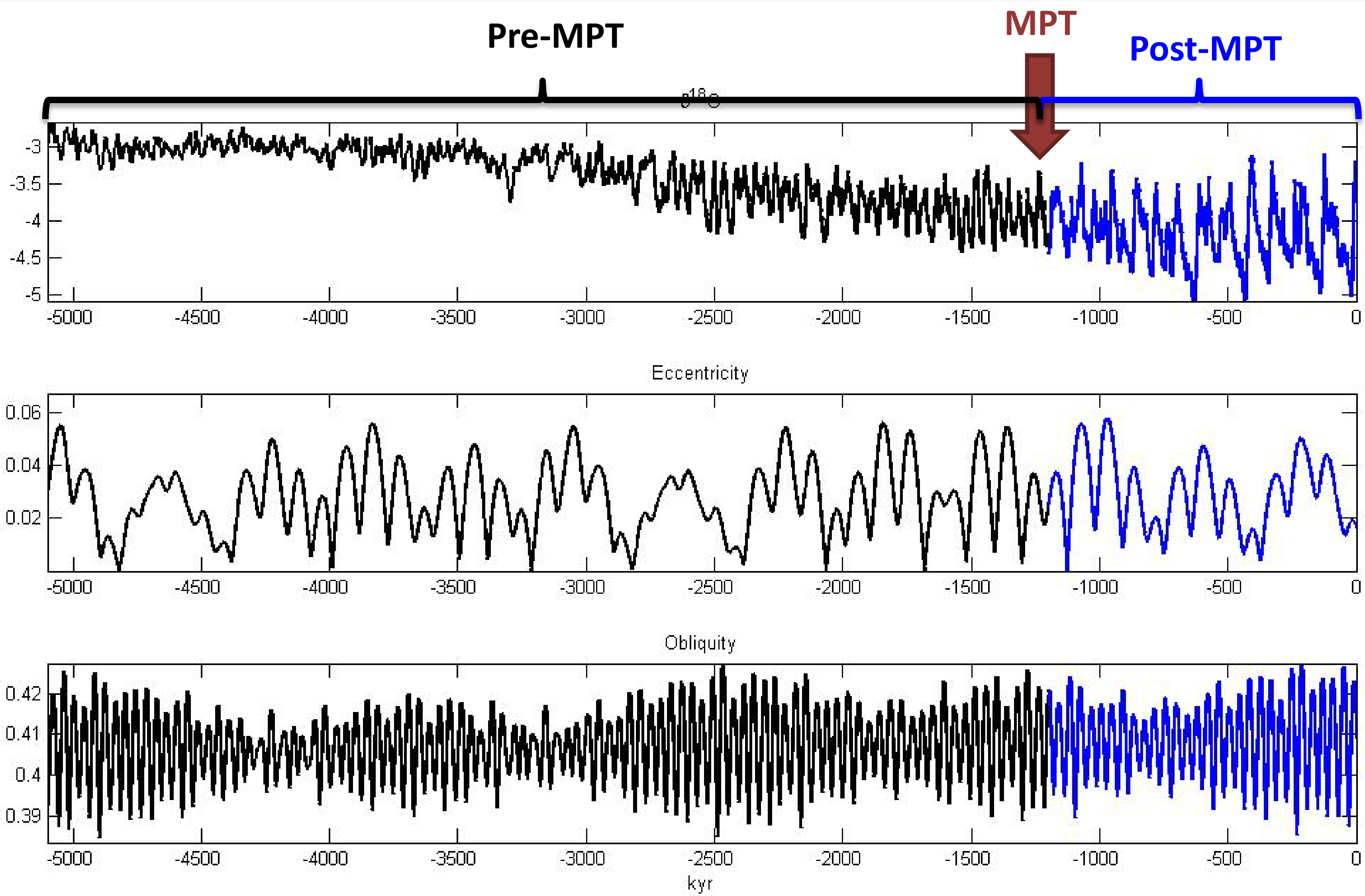


Table of Models

Identifying which type of model is the best choice for modeling the MPT is just as important as the actual fit of the model. The underlying mathematical structure might teach us something about the underlying drivers of the system.

MPT modeling options:

1. Dynamic Hopf Bifurcations
2. Relaxation Oscillators
3. Threshold/bursting models
4. Excitable System with Slow Manifold

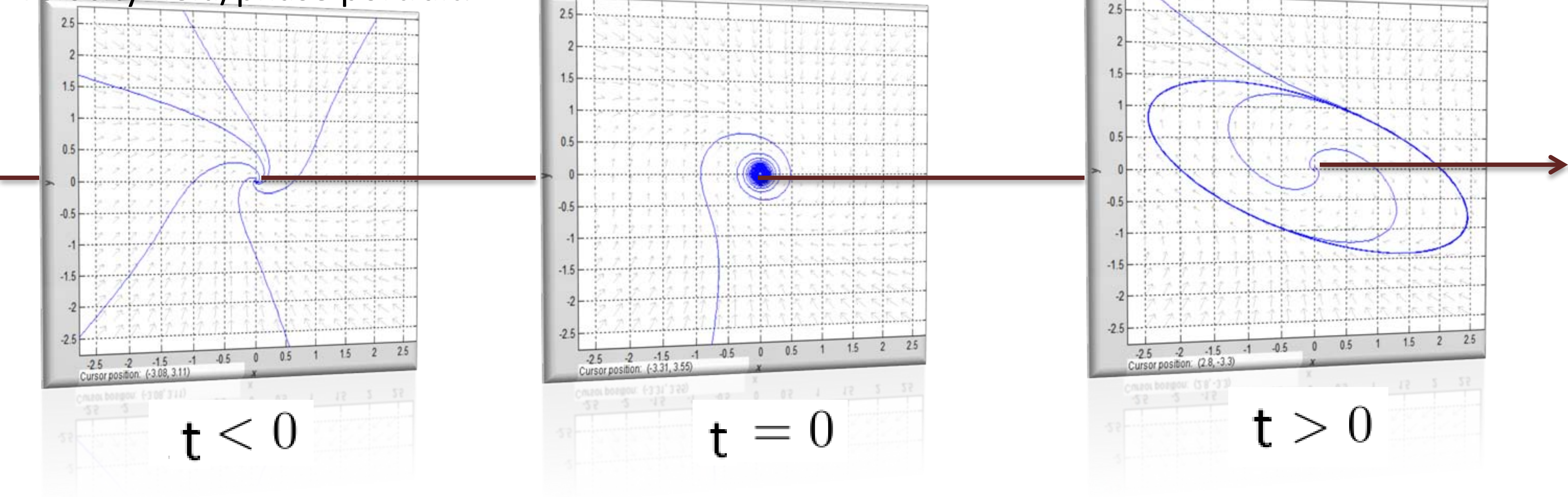
We will focus on the Dynamic Hopf Bifurcation as possible tool to uncover the secrets of the 100,000 year Problem. We will not consider methods 2-4.[1]

[1]To learn more about these systems I recommend Michel Crucifix's "Oscillators and relaxation phenomena in Pleistocene climate theory" 2012.

Dynamic Hopf Bifurcation

$$\dot{x} = y + \mu x - xy^2$$
$$\dot{y} = \text{t}\, y - x - y^3$$

Velocity field/phase portrait:



origin is a center.

origin is globally stable or a global attractor

origin is locally unstable with a globally stable limit cycle

Dynamic Hopf Bifurcation

$$\dot{x} = y + \mu x - x y^2$$
$$\dot{y} = \mathrm{t}\, y - x - y^3$$

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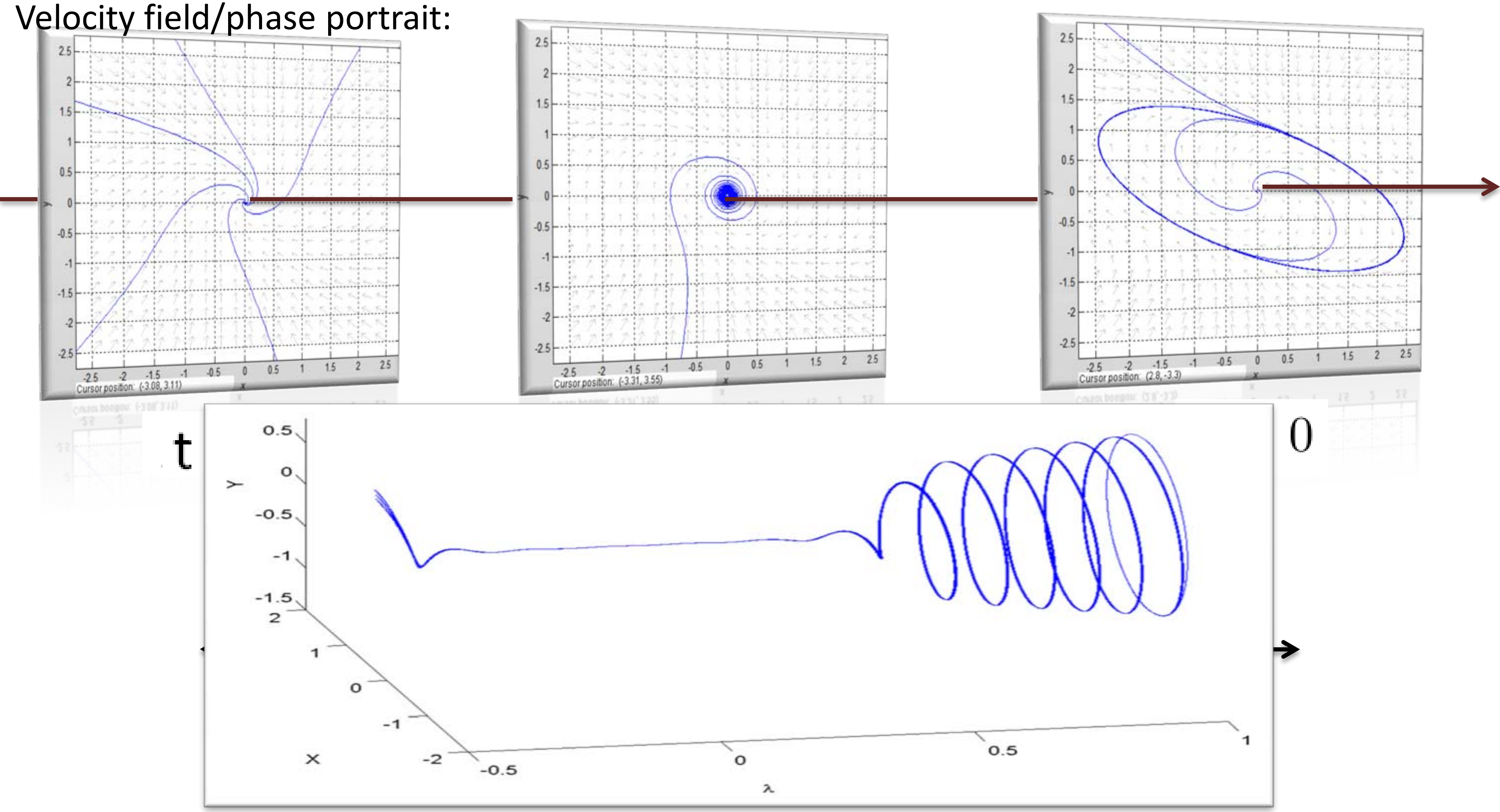
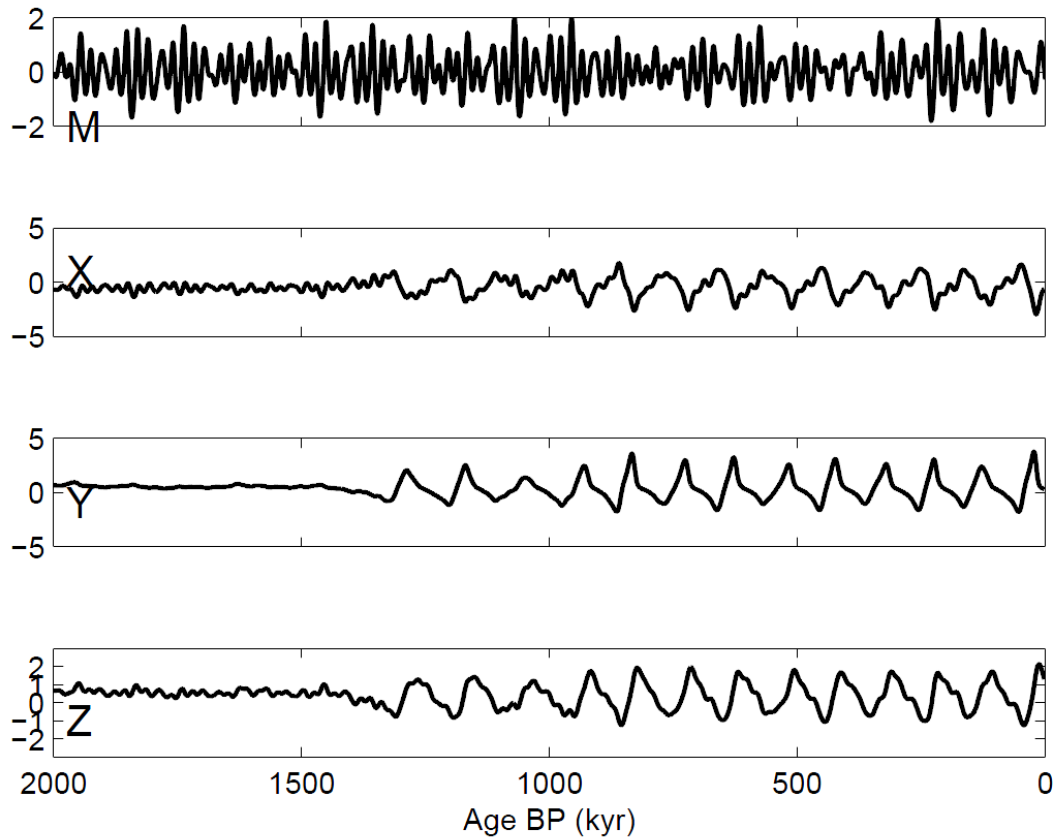


Maasch & Saltzman [1990]

Ice Line	$\dot{X}$	$=$	$-X - Y - uM(t)$
Atmospheric CO_2	$\dot{Y}$	$=$	$-pZ + rY + sZ^2 - Z^2Y$
North Atlantic Deep Water Formation	$\dot{Z}$	$=$	$-q(X + Z).$



Maasch & Saltzman [1990]

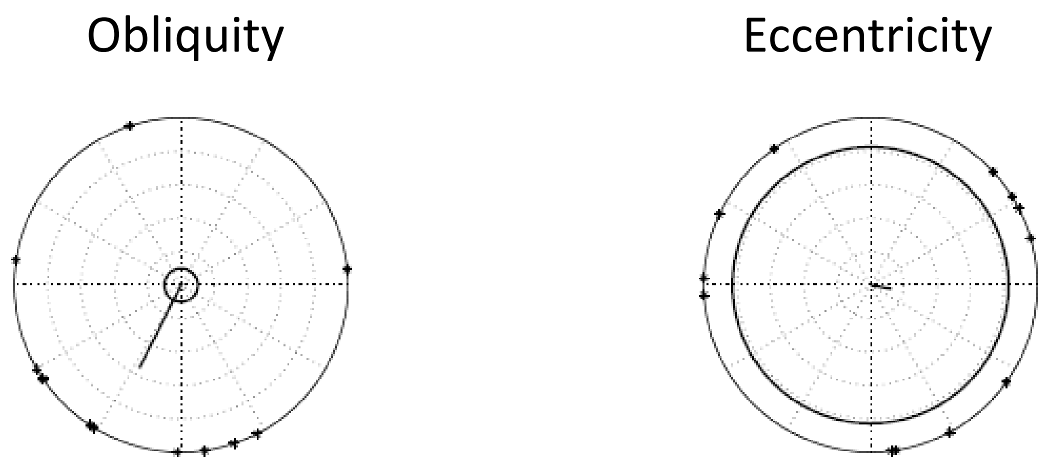
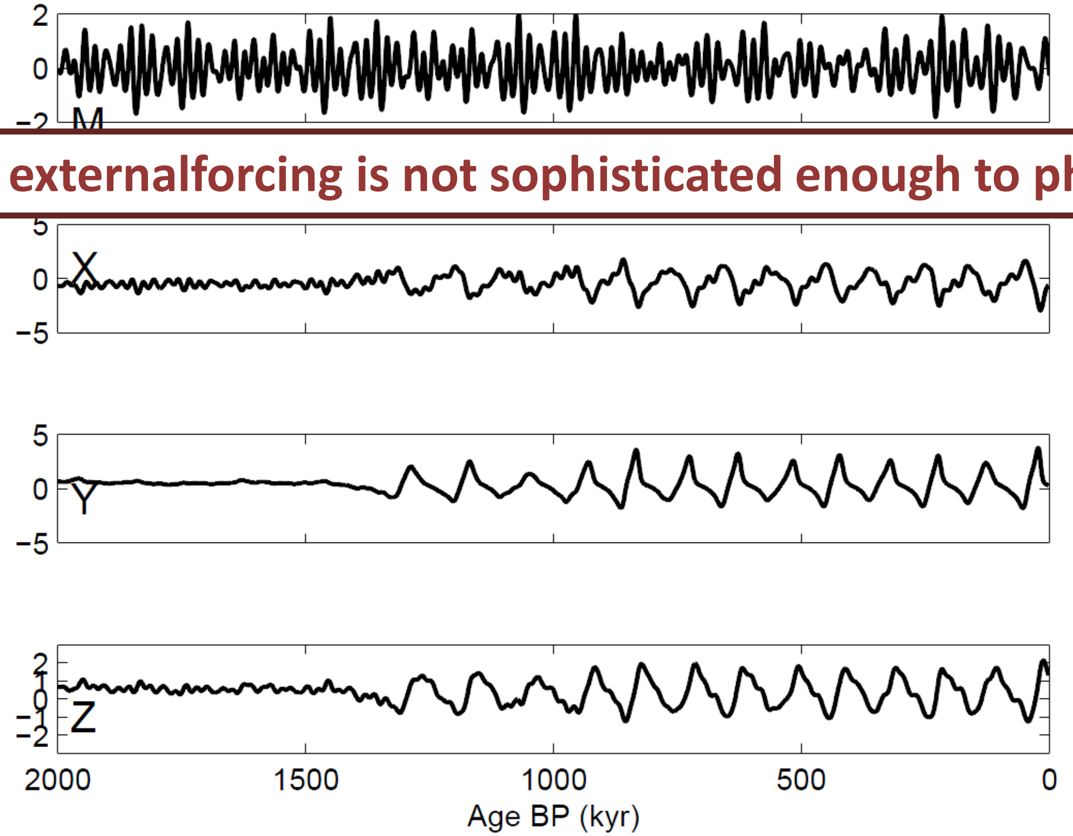


Figure 3.2: Circular statistics of the last 1.2 million years. Obliquity phase angle is on the left, eccentricity is on the right. Radial line shows mean angle with magnitude R showing relative cohesion of angles. The inner solid circle is the magnitude R must be exceed to reject H_0. Stars along the outer circle are individual phase angle differences. A radial line pointing straight up would show the model is in phase with the local maxima of the forcing term. A radial line pointing straight down would show the model is in phase with the local minima of the forcing term.

Dynamic Hopf Bifurcation

Ice Line	$\dot{X} = -X - Y - \boxed{uM(t)}$
Atmospheric CO_2	$\dot{Y} = -pZ + rY + sZ^2 - Z^2 Y$
North Atlantic Deep Water Formation	$\dot{Z} = -q(X + Z).$



Maybe the externalforcing is not sophisticated enough to phase correlate?

Improving Maasch and Saltzman

We incorporate the Budyko-Sellers-Widiasih Model into the Maasch & Saltzman model:

$$
\begin{aligned}
\dot{\eta} &= \epsilon\left(\left(\frac{C(\omega)Q(\varepsilon)(\alpha_2-\alpha_1)}{B(B+C(\omega))}\right)\left(\eta-\frac{s_2(\beta)\eta}{2}\right)+\frac{Q(\varepsilon)}{B}(1-\alpha_0)\right)- \\
&\quad \left(\frac{A(\mu)}{B}+\left(\frac{1-s_2(\beta)\eta^3}{2}\right)\frac{C(\omega)Q(\varepsilon)(\alpha_2-\alpha_1)}{B(B+C(\omega))}\right)+\left(\frac{Q(\varepsilon)s_2(\beta)(1-\alpha_0)}{2(B+C(\omega))}(3\eta^2-1)\right)-T_c \\
\dot{\mu} &= -p\omega+r\mu+s\omega^2-m\omega^2\mu \qquad (4.1) \\
\dot{\omega} &= -q(f(\eta)+\omega)
\end{aligned}
$$

$$
\begin{array}{llllll}
\alpha_1 &= 0.32 & p &= 0.8 & A(\mu) &= -3\mu+205.5 \\
\alpha_2 &= 0.62 & q &= 1.8 & C(\omega) &= 0.05\omega+3 \\
\alpha_0 &= (\alpha_1+\alpha_2)/2 & m &= 1 & f(\eta) &= -16\eta+13.2 \\
B &= 1.9 & T_c &= -10 & s_2 &= (5/16)(-2+3\sin\beta^2)
\end{array}
$$

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&\quad \left(\frac{A(\mu)}{B}+\left(\frac{1-s_2(\beta)\eta^3}{2}\right)\frac{C(\omega)Q(\varepsilon)(\alpha_2-\alpha_1)}{B(B+C(\omega))}\right)+\left(\frac{Q(\varepsilon)s_2(\beta)(1-\alpha_0)}{2(B+C(\omega))}(3\eta^2-1)\right)-T_c \\
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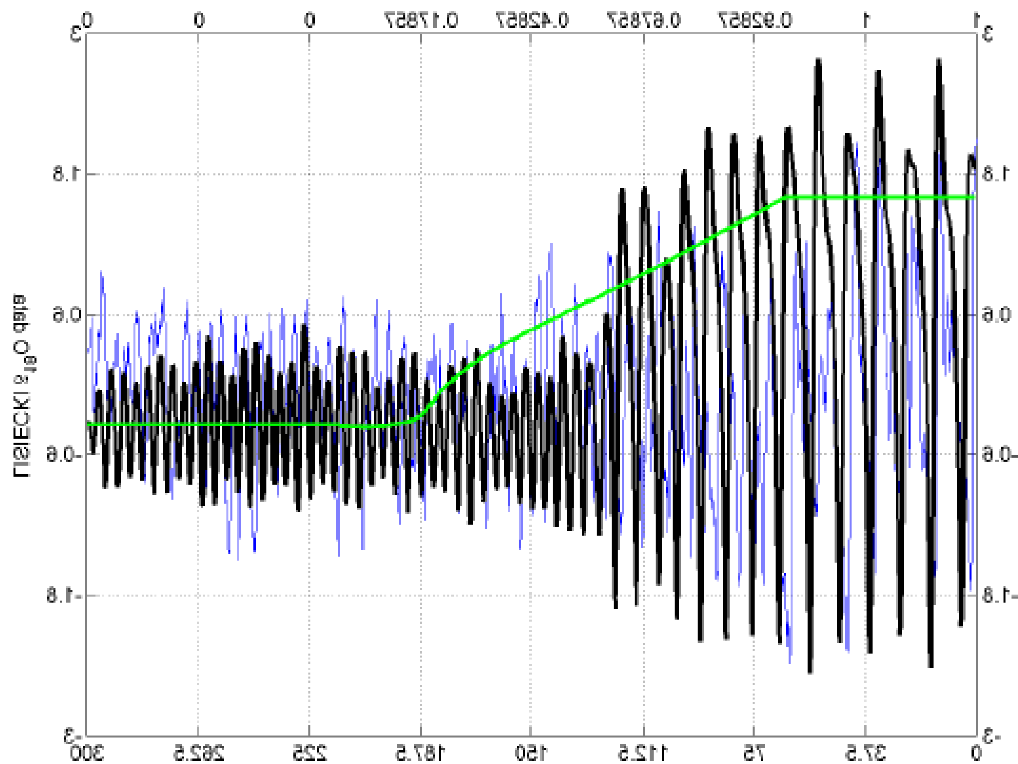
$$



Dynamic Hopf Bifurcation

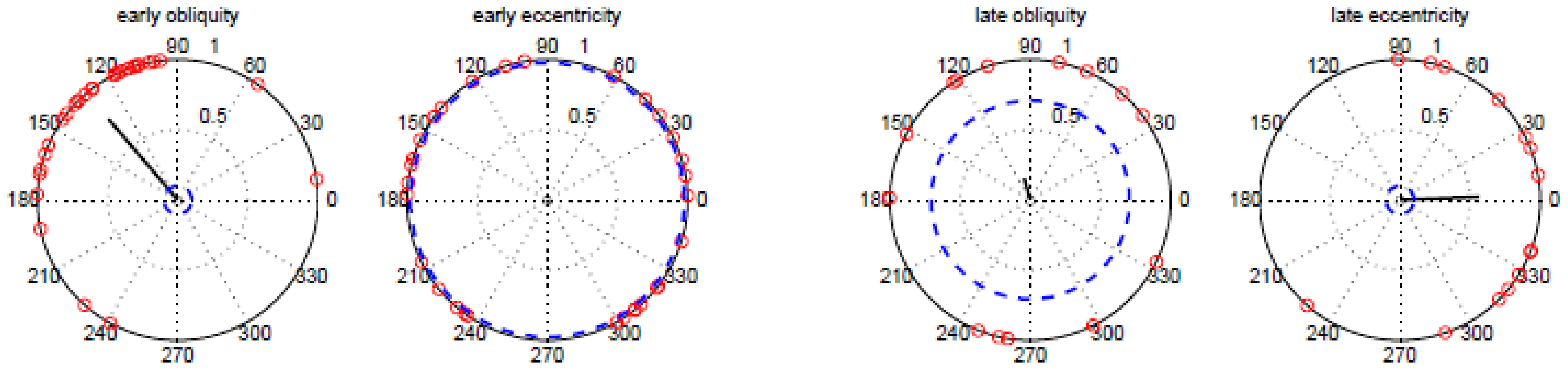
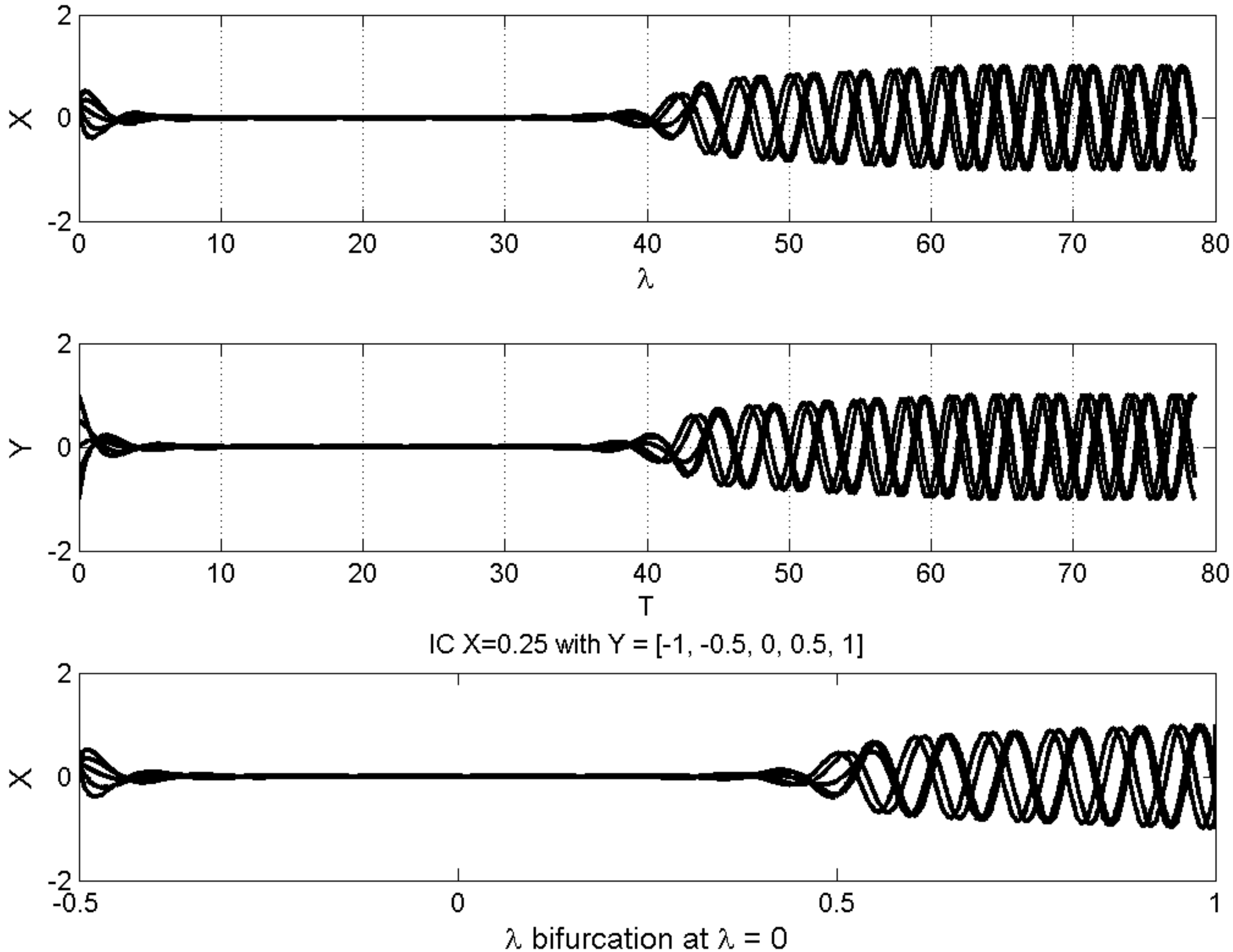


Figure 4.3: Phase angle analysis between model ice line output and orbital forcing for early (3 My - 1.2 My) and late (1.2 My to present) Pleistocene. The reader is referenced to Zar (1999) and Upton and Fingleton (1989) for details on the circular statistics used to produce these diagrams [31, 32]. Lisiecki (2010) also presents a concise review of the process [8].

Dynamic Hopf Bifurcation

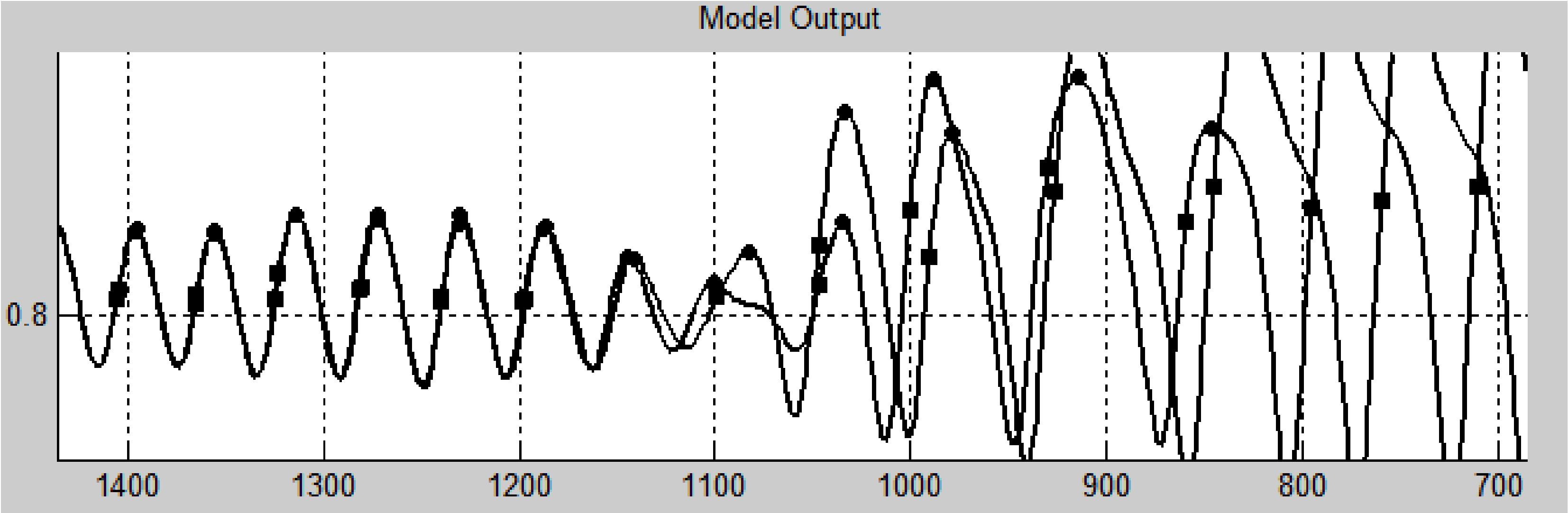
And it got worse . . .

Initial Conditions of Dynamic Hopf



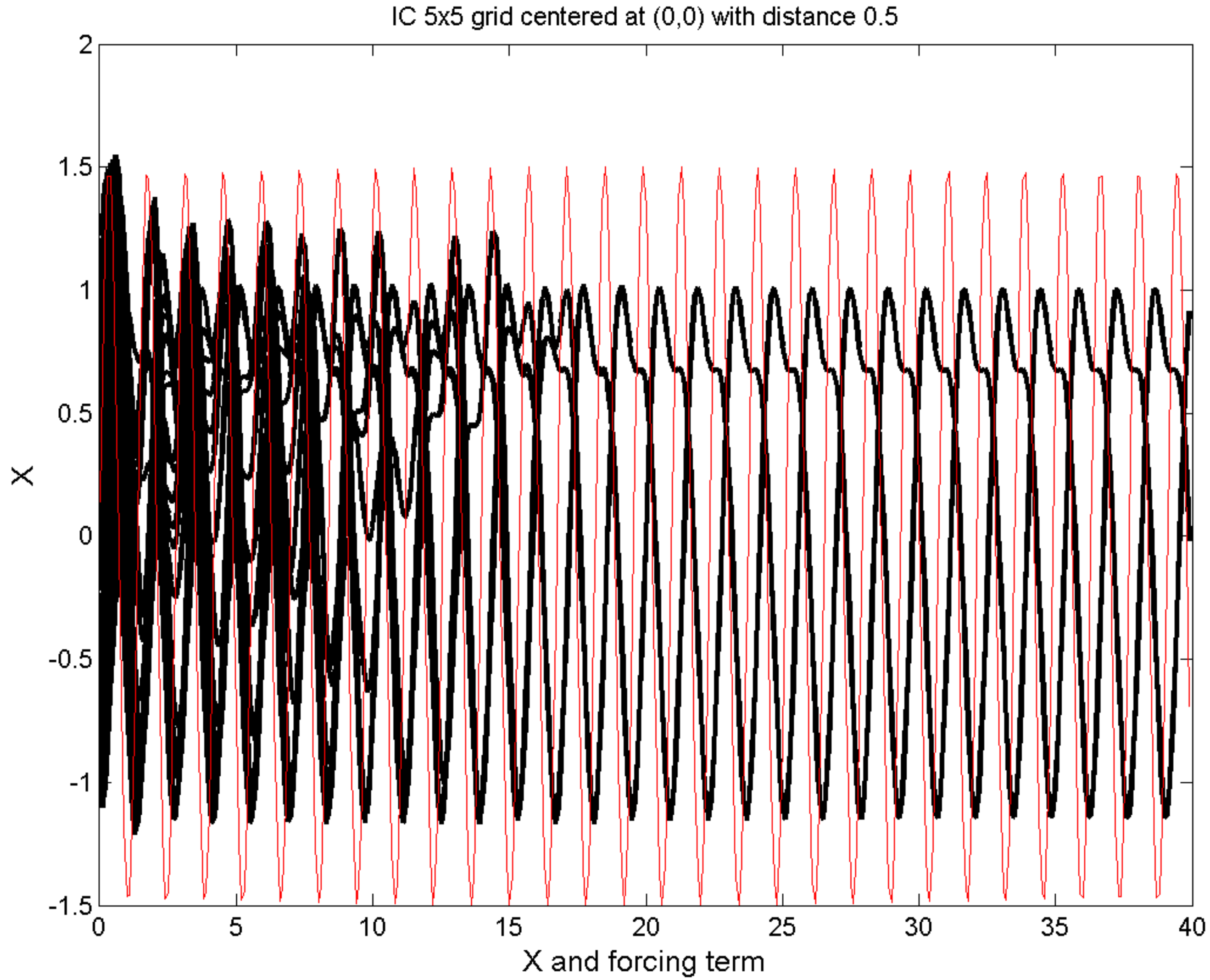
Sensitivity to Initial Conditions

Initial Conditions of Dynamic Hopf



One example of Sensitivity to Initial Condition at the neck.

Initial Conditions of Dynamic Hopf



Order in Chaos: phenomena of stable trajectories in a dynamical system.

What comes next?

There are complicated issues happening in this model. How do we understand this mathematically?

By reducing the complexity of the problem we will highlight the real connection between the phase angle of the model and the forcing.

Next up: a generalized forced oscillator with a dynamic Hopf bifurcation.

Goal: To show phase correlation is unpredictable and ubiquitous to all dynamic Hopf bifurcation models.

What comes next?

To do a general study of Hopf bifurcation on a periodically forced oscillator, we will analyze an equivalent system.

We analyze a periodically forced Hopf bifurcation of maps which is tractable and can be analyzed with computer.

We use the principle first employed in:

Richard P. McGehee and Bruce B. Peckham. Resonance Surfaces for Forced Oscillators. *Experimental Mathematics*, 3(3):221–244, 1994.

McGehee Peckham Model

$H_{(\alpha,\beta)} = g_\alpha \circ h_\beta$ as follows:

$$h_\beta = \begin{cases} r_{i+1} = (r_i(1 - r_i^2))/(1 + r_i^2) \\ \theta_{i+1} = 2\pi\beta + (1 - r_i^2)/(1 + r_i^2) \end{cases}$$

$$g_\alpha = \begin{cases} x_{i+1} = \alpha(1 - x_i) \\ y_{i+1} = -\alpha y_i \end{cases}$$

This is a discrete period map. That is, the discrete steps move the system forward exactly one period whose simulated period is based on

Note: The *h* equation is in Polar coordinates while the *g* equation is in Cartesian coordinates. This is done for clarity of communicating what each map does.

Richard P. McGehee and Bruce B. Peckham. Resonance Surfaces for Forced Oscillators. *Experimental Mathematics*, 3(3):221–244, 1994.

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$$g_\alpha = \begin{cases} x_{i+1} = \alpha(1 - x_i) \\ y_{i+1} = -\alpha y_i \end{cases}$$

For alpha = 0:

The unit circle is invariant with a rigid rotation with rotation number beta.

For 1 >> alpha>0:

The circle distorts but remains invariant.

For alpha near 1:

There is only an attracting fixed point at (x,y) = (1,0).

Beta:

The rotation number of the invariant circle.

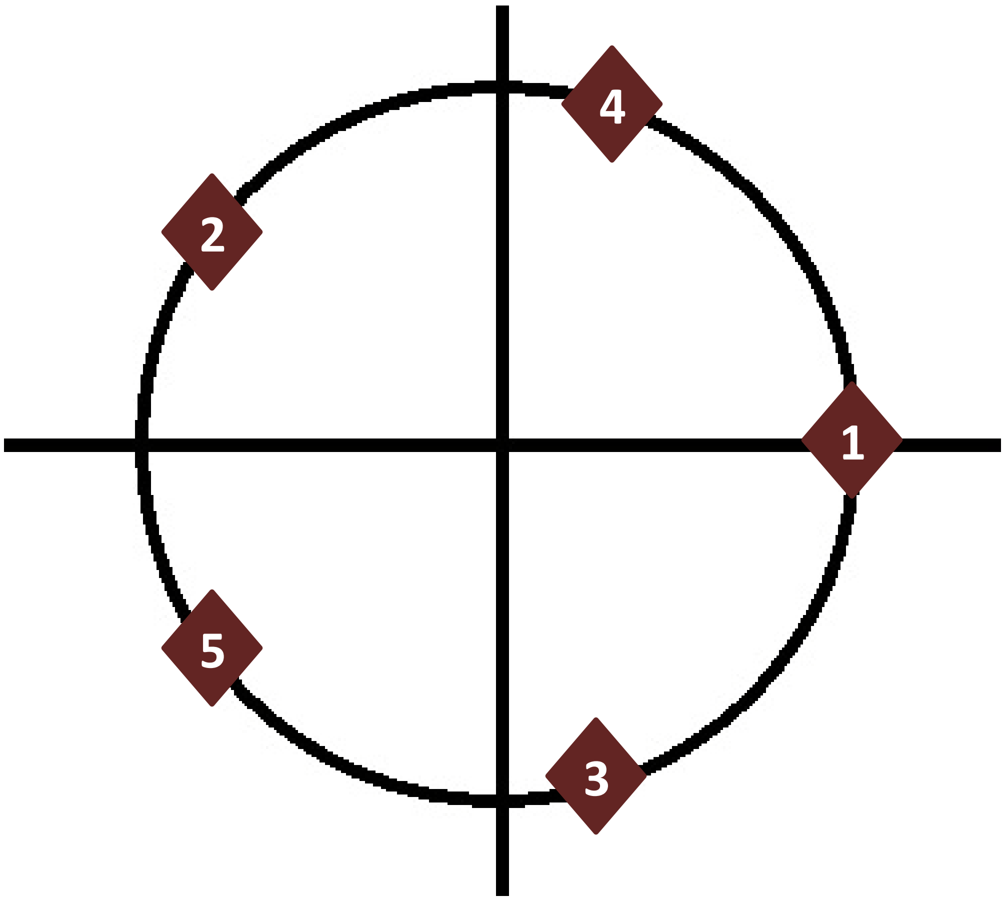
McGehee Peckham Model

Our planet has a period of 100 kyr and obliquity has a period of 41 kyr.

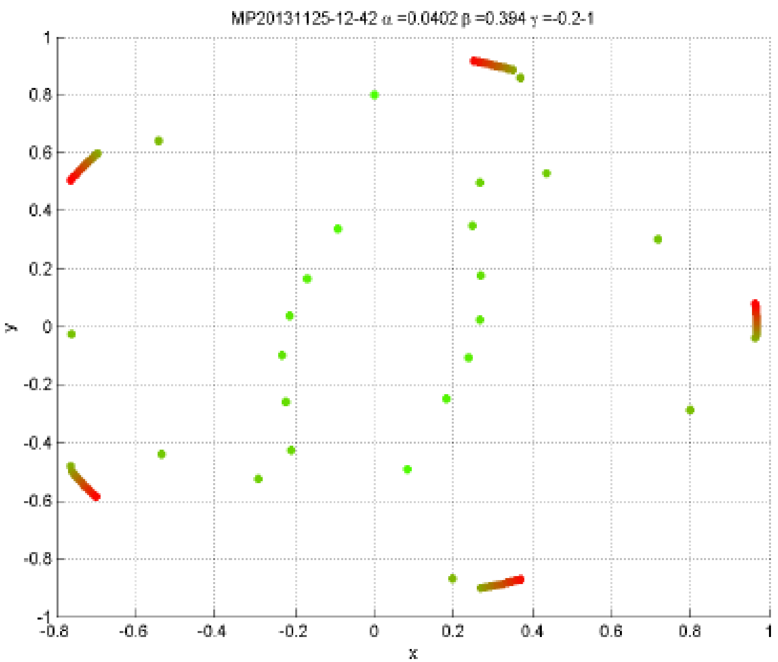
We chose to only consider one external forcing to reduce the complexity of the forcing from quasi-periodic to periodic.

Obliquity is chosen because it is the most dominant influence on the planet of the external forcing terms.

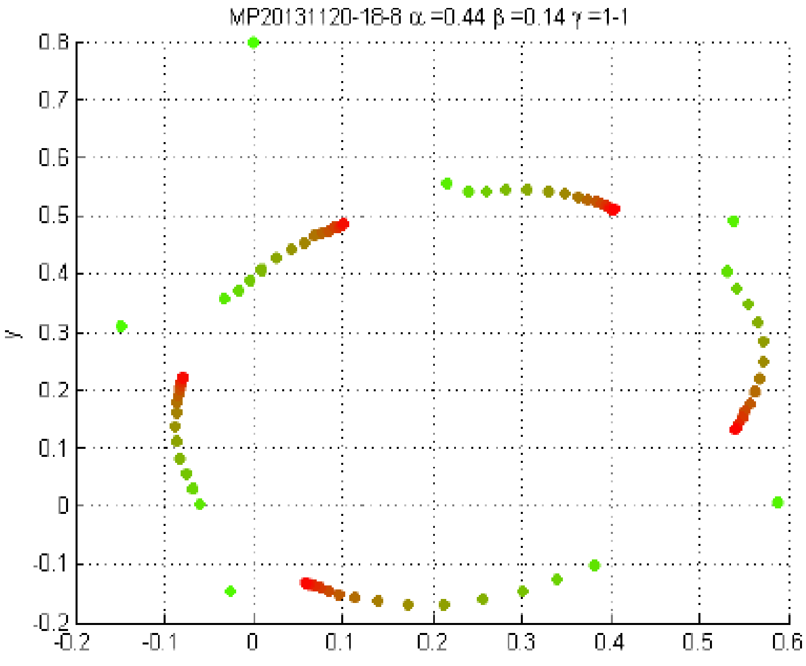
We expect a planetary resonance of 4:10 (or 2:5). So in phase space:



McGehee Peckham Model



$p_1 = (0.04, 0.39)$



$p_2 = (0.44, 0.14)$

By plotting all time steps on the phase place, one can determine if the trajectory converge to a finite number of point(s) in the phase plane. Each circle represents a 40 kyr simulated time step.

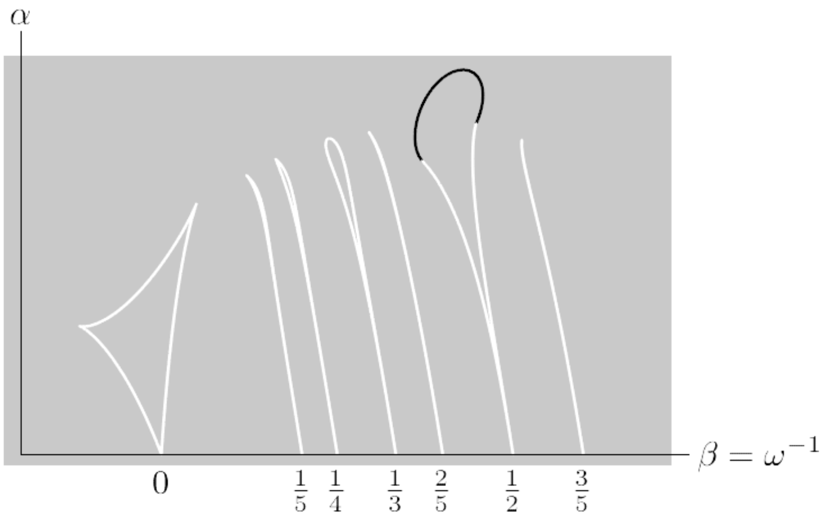
If so, then the trajectory has a stable cycle that repeats every 40,000**n** simulated years, where **n** is the number of distinct clusters of trajectories on the phase place.

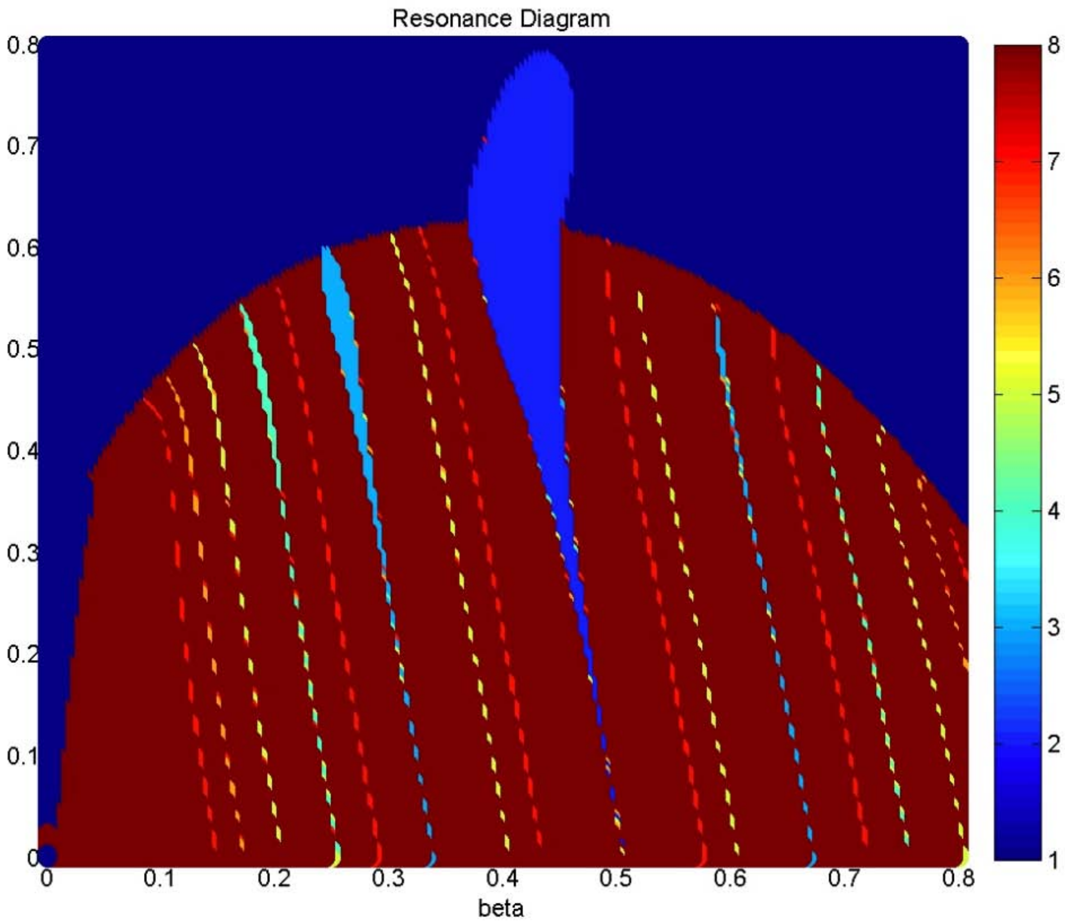
Earlier times are colored green and later times are colored red.

McGehee Peckham Model

By testing each point in (α, β) space, an Arnold Tongue diagram can be produced for the parameter plane.

Then and Now





But this map cannot simulate dynamic bifurcations… Yet.

Revised McGehee Peckham Model

$$G_{(\alpha,\beta,\gamma)} = g_\alpha \circ h_\beta \circ k_\gamma$$

$$k_\gamma = \quad r_{i+1} = (\gamma - 1) r_i$$

$$h_\beta = \begin{cases} r_{i+1} = (r_i(1 - r_i^2))/(1 + r_i^2) \\ \theta_{i+1} = 2\pi\beta + (1 - r_i^2)/(1 + r_i^2) \end{cases}$$

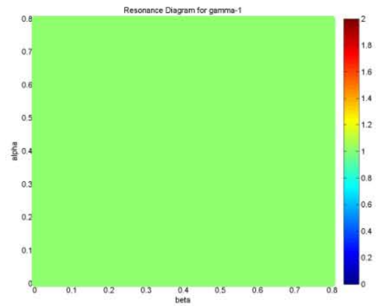
$$g_\alpha = \begin{cases} x_{i+1} = \alpha(1 - x_i) \\ y_{i+1} = -\alpha y_i \end{cases}$$

α = Forcing amplitude
β = Ratio of natural frequency to forcing frequency
γ = Bifurcation Parameter

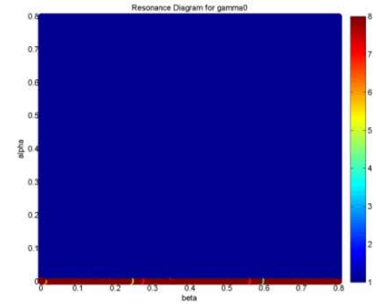
We can now create Arnold Tongue diagrams for:

$$\gamma \in \{-0.1, 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1\}$$

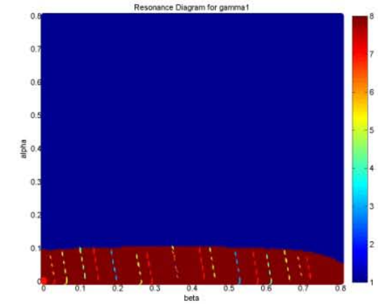
McGehee



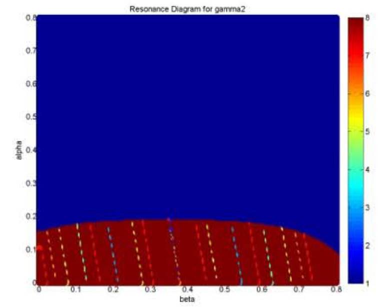
gamma=- 0.1



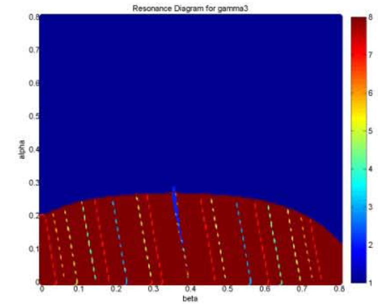
gamma =0



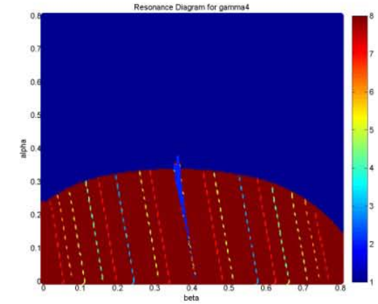
gamma = 0.1



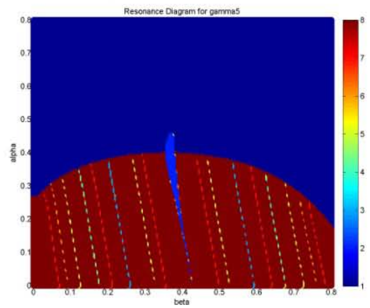
gamma = 0.2



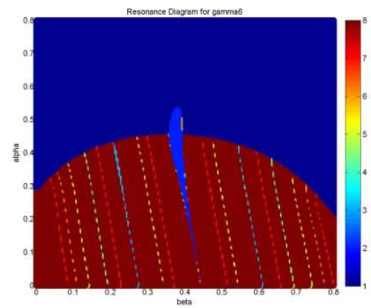
gamma = 0.3



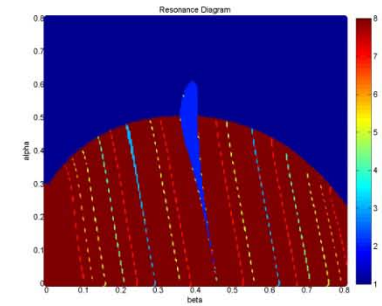
gamma = 0.4



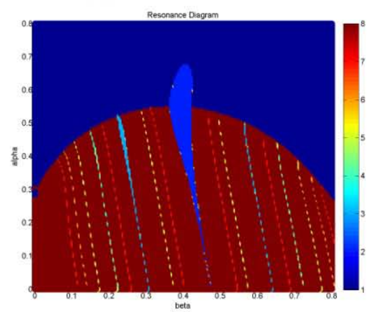
gamma = 0.5



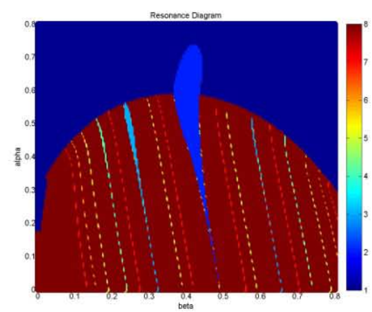
gamma = 0.6



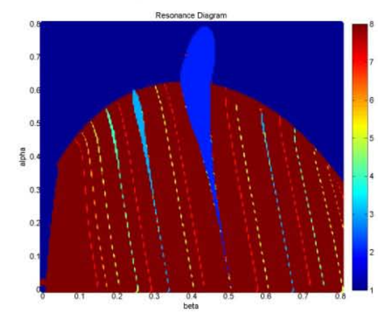
gamma=0.7



gamma = 0.8

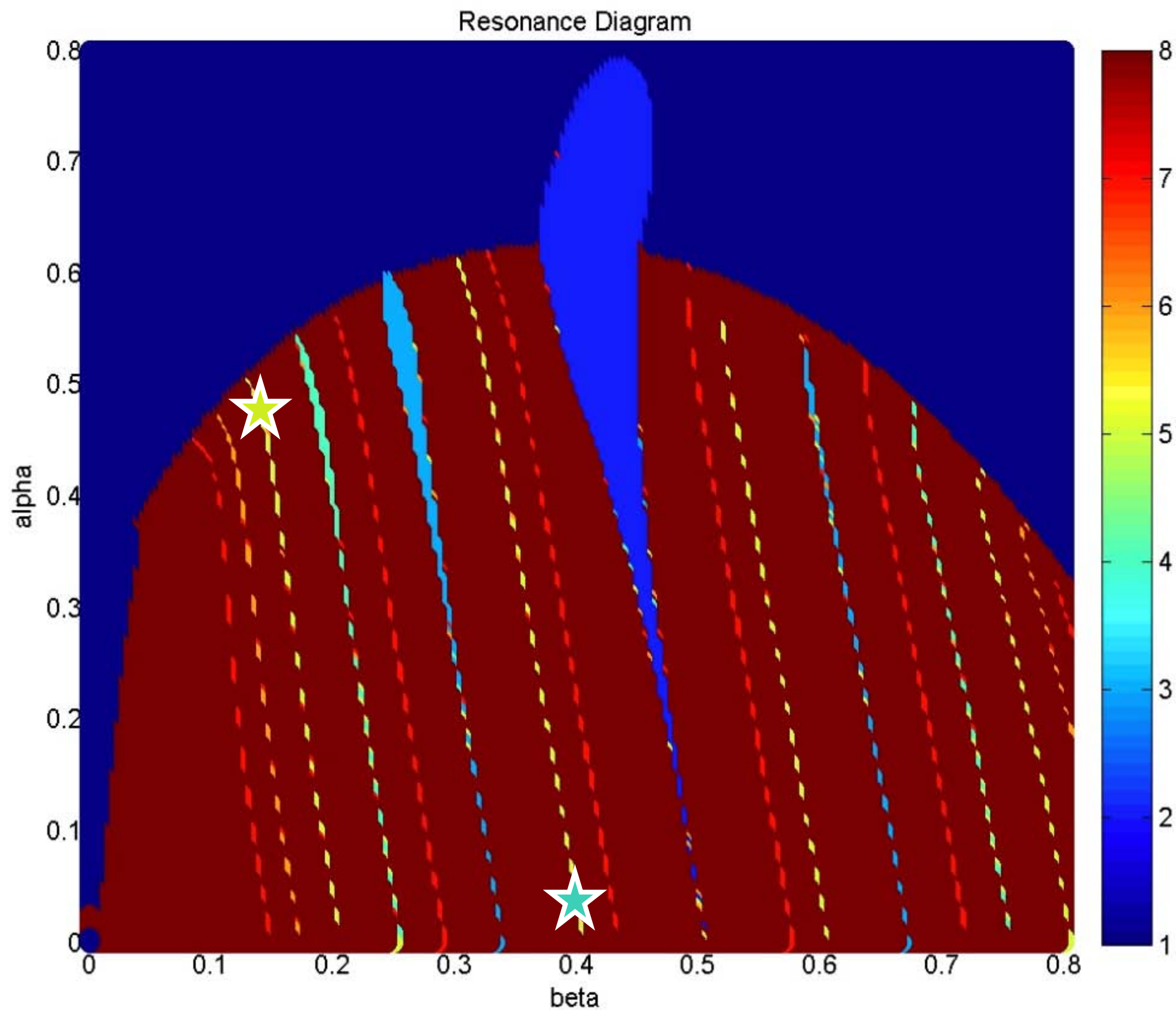


gamma = 0.9



gamma = 1

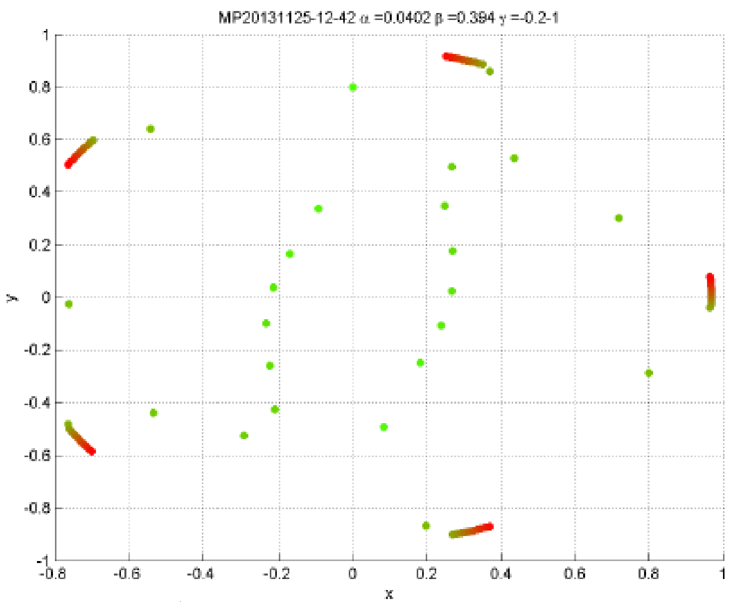
McGehee Peckham Model



$p_1 = (0.04, 0.39)$

$p_2 = (0.44, 0.14)$

McGehee Peckham Model



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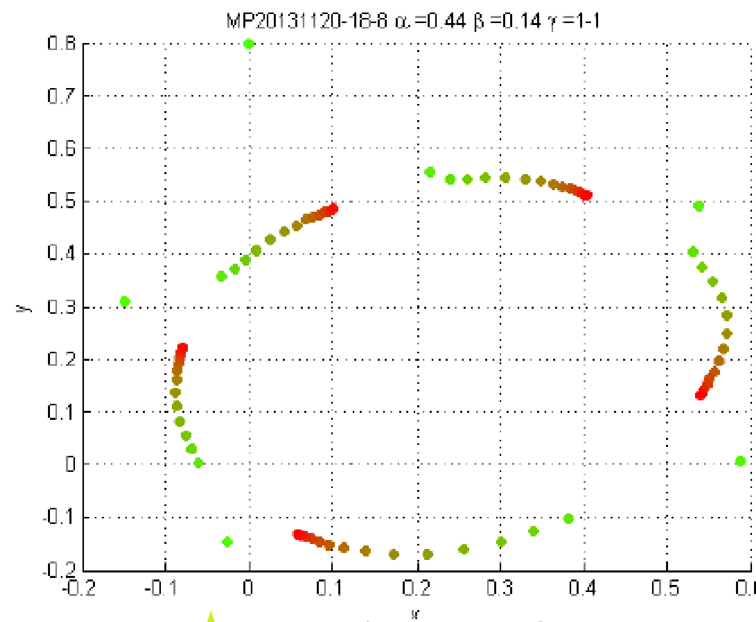
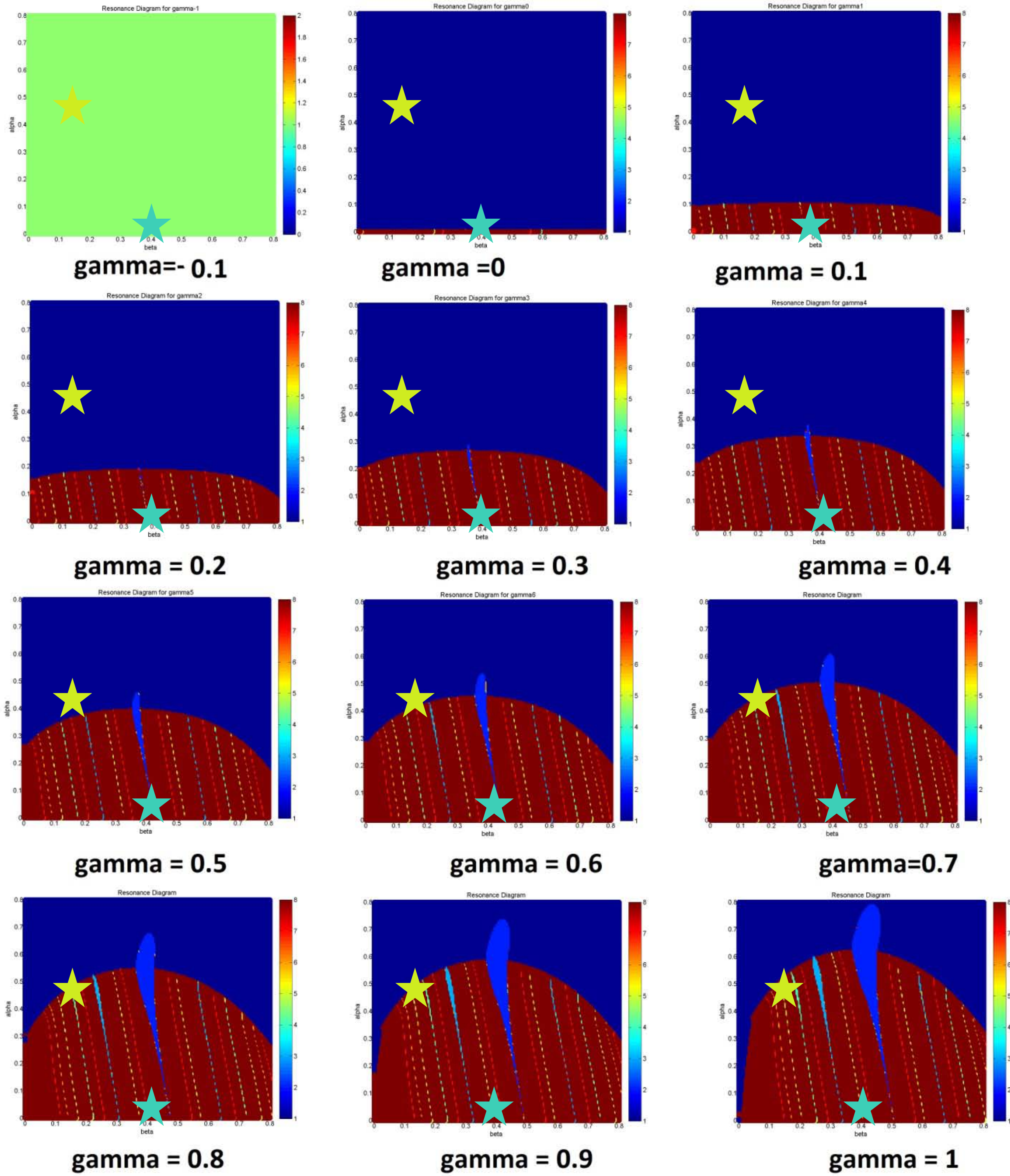


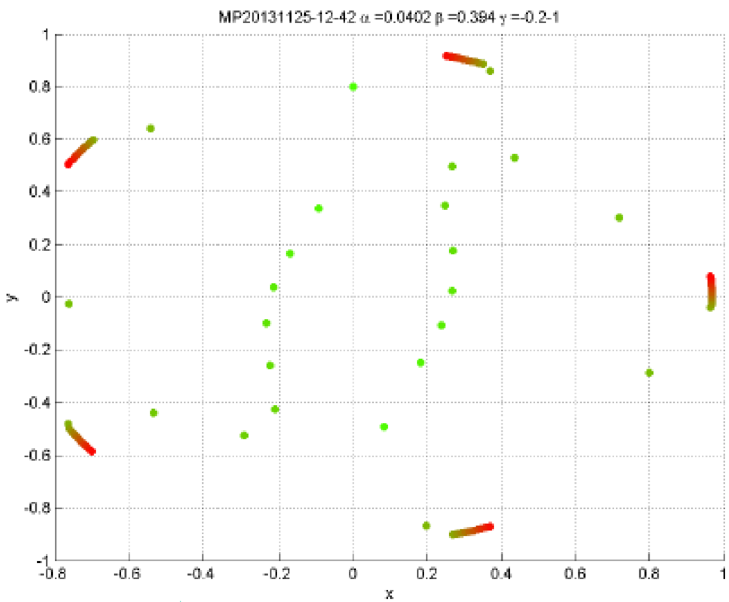
$p_2 = (0.44, 0.14)$

Phase portrait of period map with □=1 for two initial conditions (0.04, 0.39) on the left and (0.44, 0.14) on the right. Earlier times are colored green and later times are colored red.

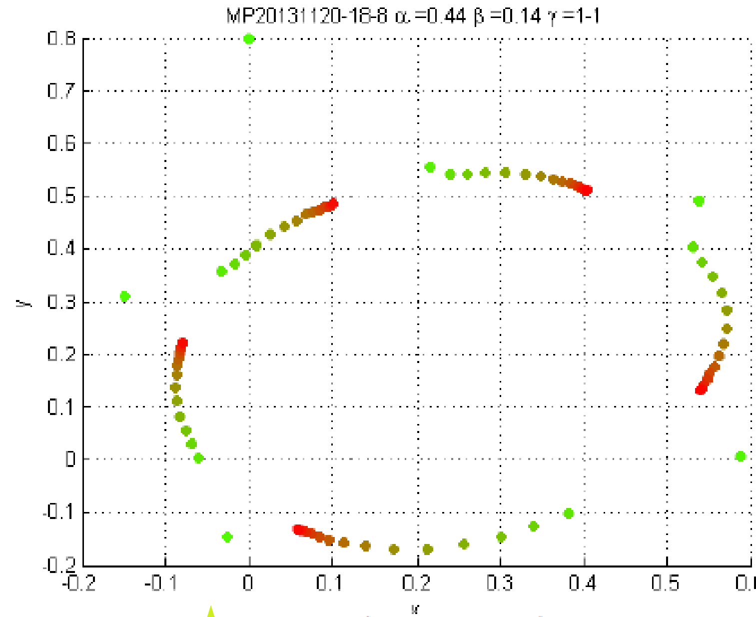
McGehee


gamma=- 0.1
gamma =0
gamma = 0.1
gamma = 0.2
gamma = 0.3
gamma = 0.4
gamma = 0.5
gamma = 0.6
gamma=0.7
gamma = 0.8
gamma = 0.9
gamma = 1

McGehee Peckham Model



$p_1 = (0.04, 0.39)$



$p_2 = (0.44, 0.14)$

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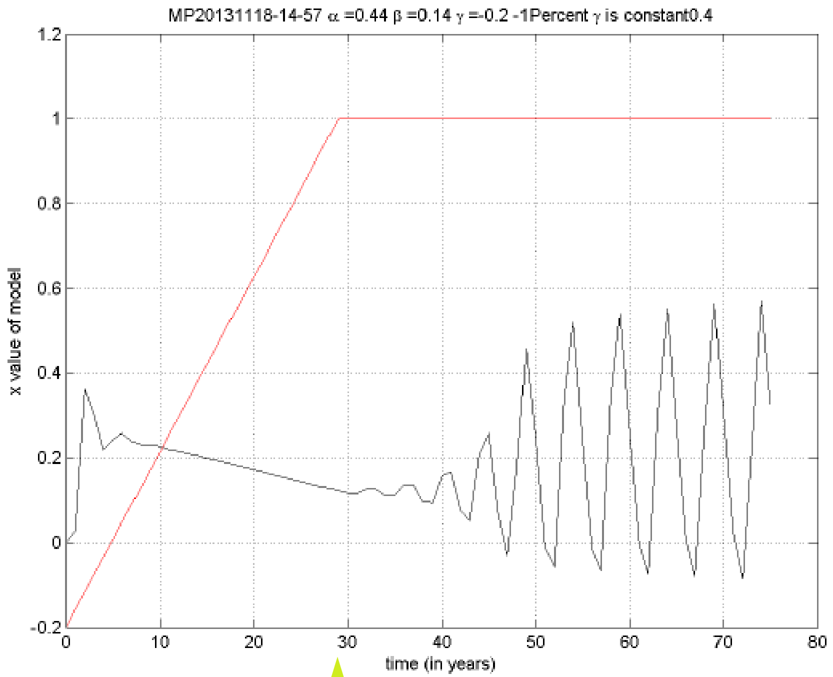
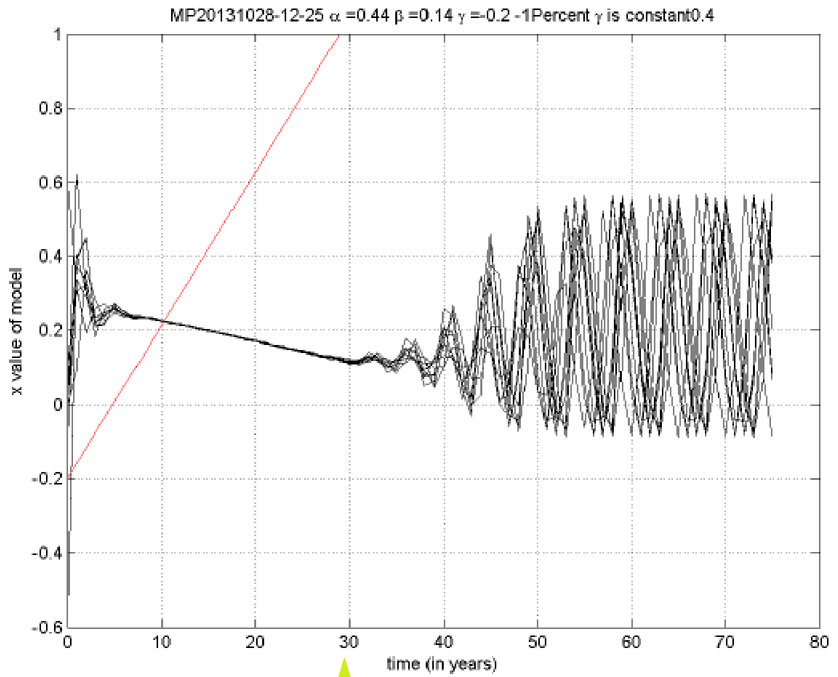
McGehee Peckham Model



$p_2 = (0.44, 0.14)$

1 initial condition



$p_2 = (0.44, 0.14)$

15 initial conditions

But this period map with 40 kyr period is not a very good way to get a time series, or phase angle statistics.

Revised McGehee Peckham

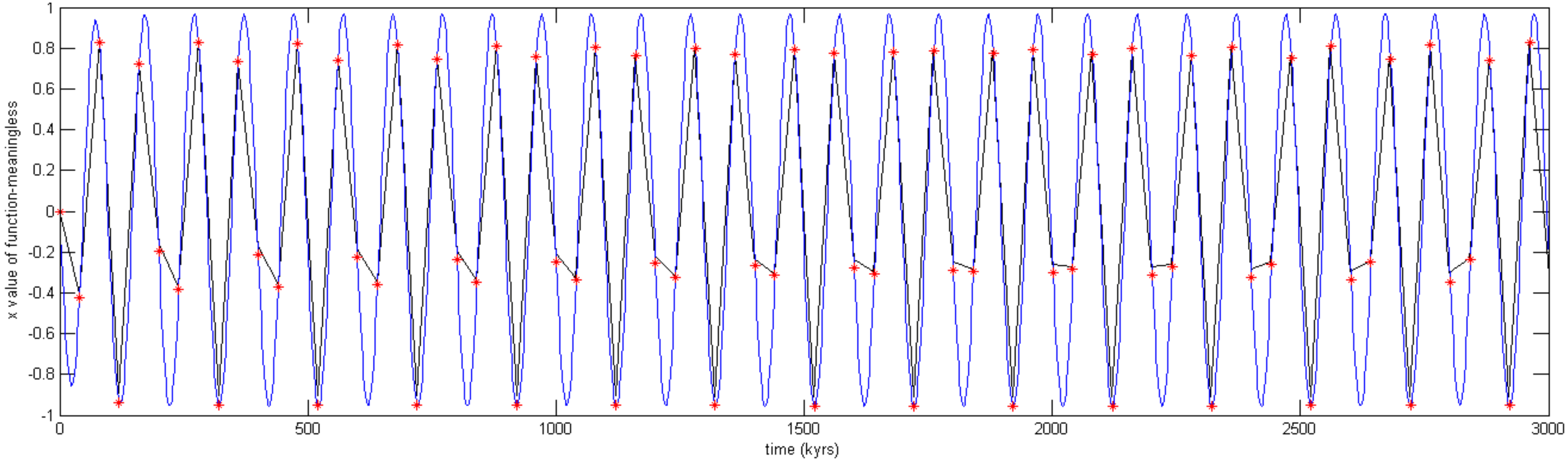
$$F_{(\alpha,\beta,\gamma,\sigma)} = f_\sigma \circ k_\gamma \circ h_\beta \circ g_\gamma$$

$$\begin{aligned}
f_\sigma &= \quad y_{i+1} = -\sigma(y_i - \mathcal{M}(t)) \\
k_\gamma &= \quad r_{i+1} = (\gamma - 1)r_i \\
h_\beta &= \begin{cases} r_{i+1} = (r_i(1 - r_i^2))/(1 + r_i^2) \\ \theta_{i+1} = 2\pi\beta + (1 - r_i^2)/(1 + r_i^2) \end{cases} \\
g_\alpha &= \begin{cases} x_{i+1} = \alpha(1 - x_i) \\ y_{i+1} = -\alpha y_i \end{cases}
\end{aligned}$$

F▯ ▯reintroduces the variation which takes place on time scales smaller than 40 kyr.

This is a period map with time step of 1 kyr instead of 40 kyr.

Revised McGehee Peckham

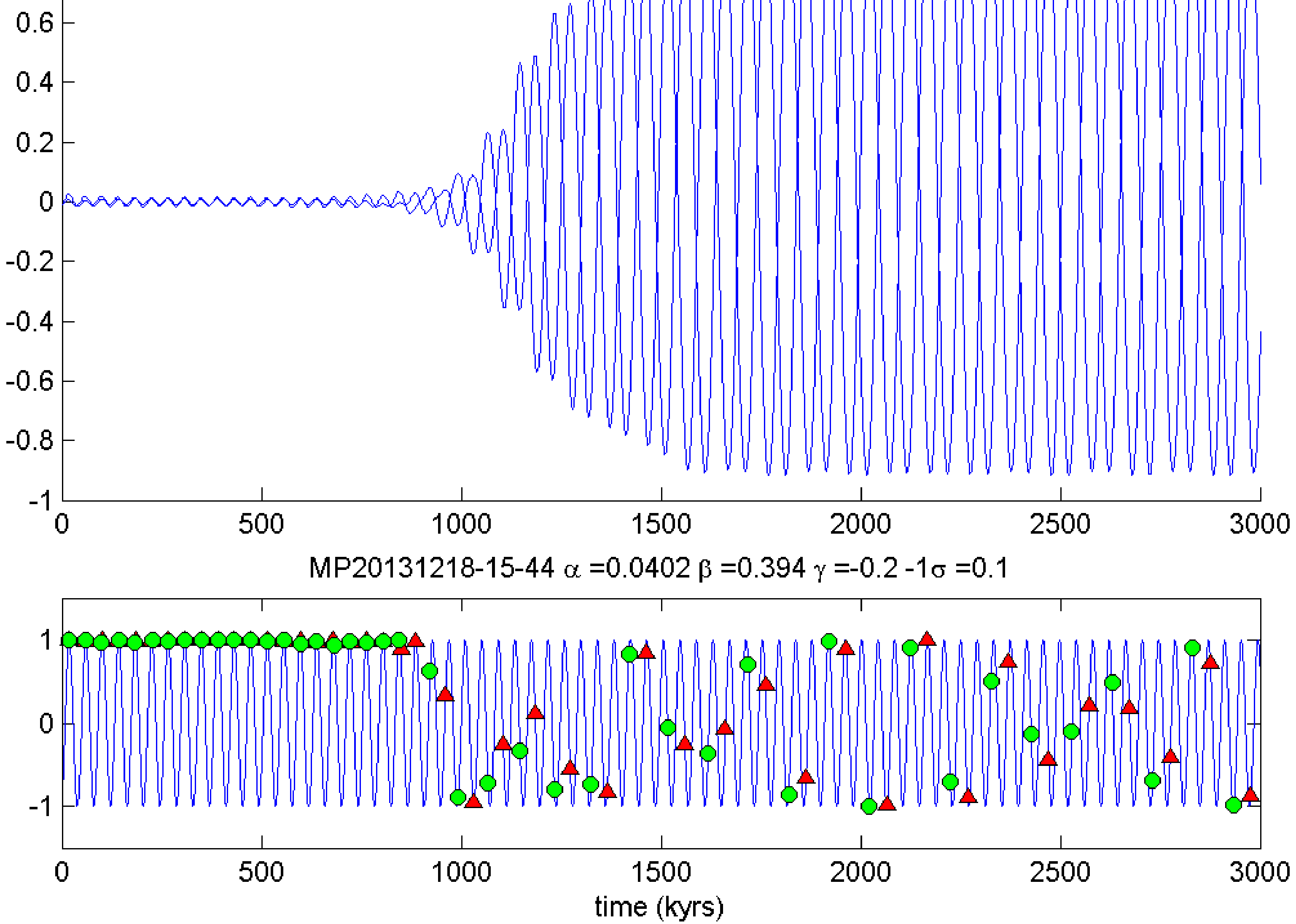


$p_1 = (0.04, 0.39)$

Revised McGehee Peckham

$T_{(-0.01,0.01)}$ and $T_{(-0.01,-0.01)}$

★ $p_1 = (0.04, 0.39)$



Stable Trajectory of a Dynamic System

Definition: A **stable trajectory** is one for which the largest Lyapunov exponent, λ_{max}, is negative. The largest Lyapunov exponent, λ_{max}, is mathematically defined as:

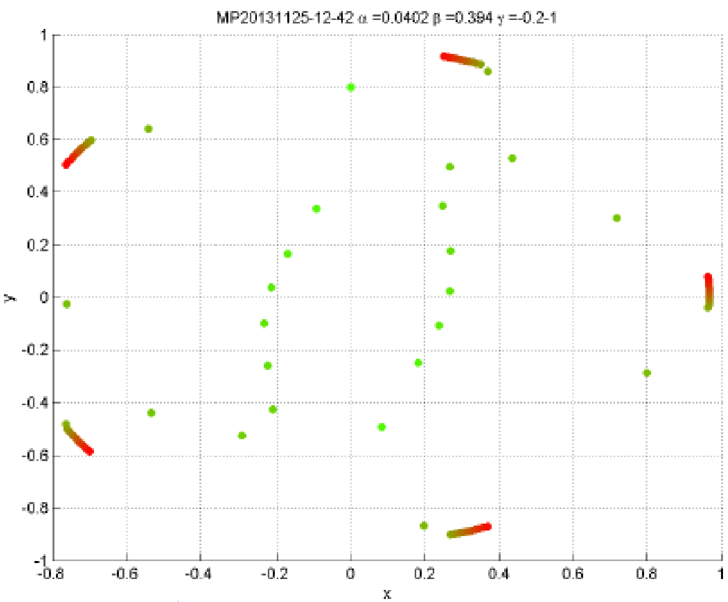
$$\lambda_{max} = \lim_{|\delta Z(0)| \to 0} \lim_{t \to \infty} \frac{1}{t} ln \frac{|\delta \mathbb{Z}(t)|}{|\delta \mathbb{Z}(0)|}$$

where δZ = [δx, δy] are vanishing perturbations around x and y.

For further reading on stable trajectories of dynamic Hopf bifurcations, see De Saedeleer (2012) and Wieczorek (2012) which offers good insights into identifying and counting stable trajectories.

Goal: To partition the domain into basins of attraction or resonance regions.

Basins of Attraction



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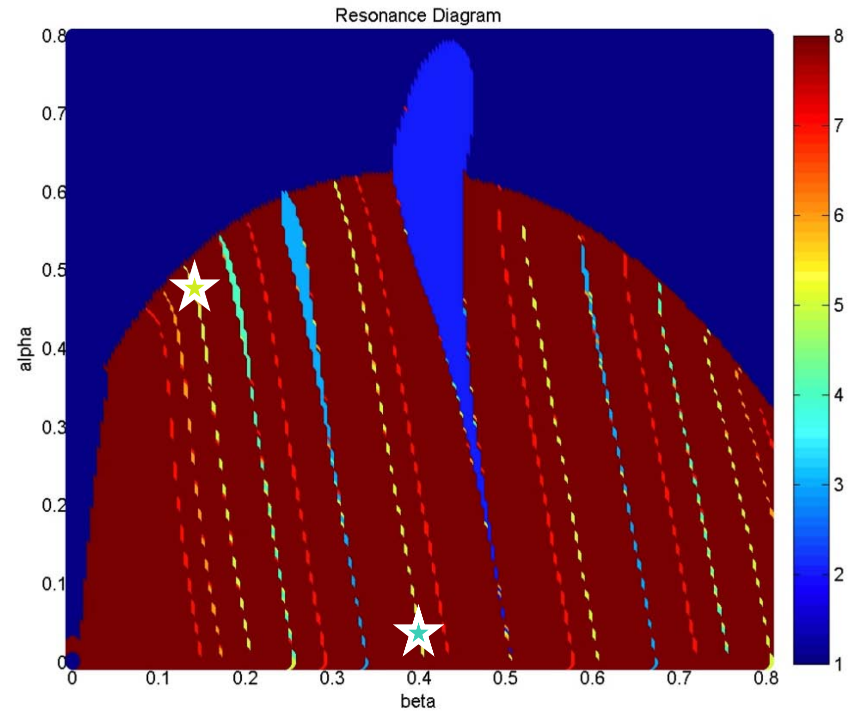
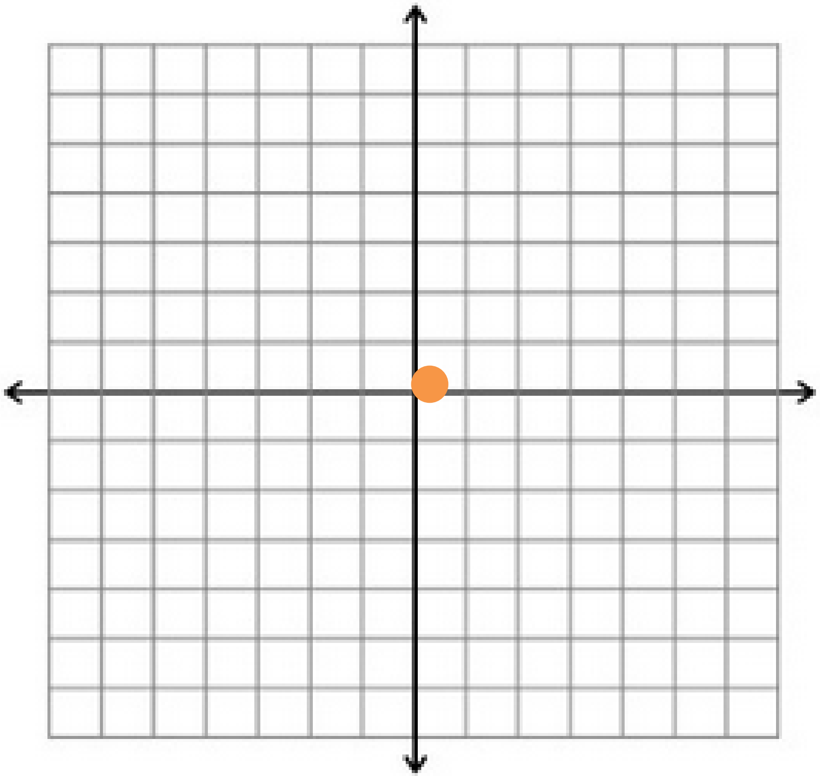


Method:

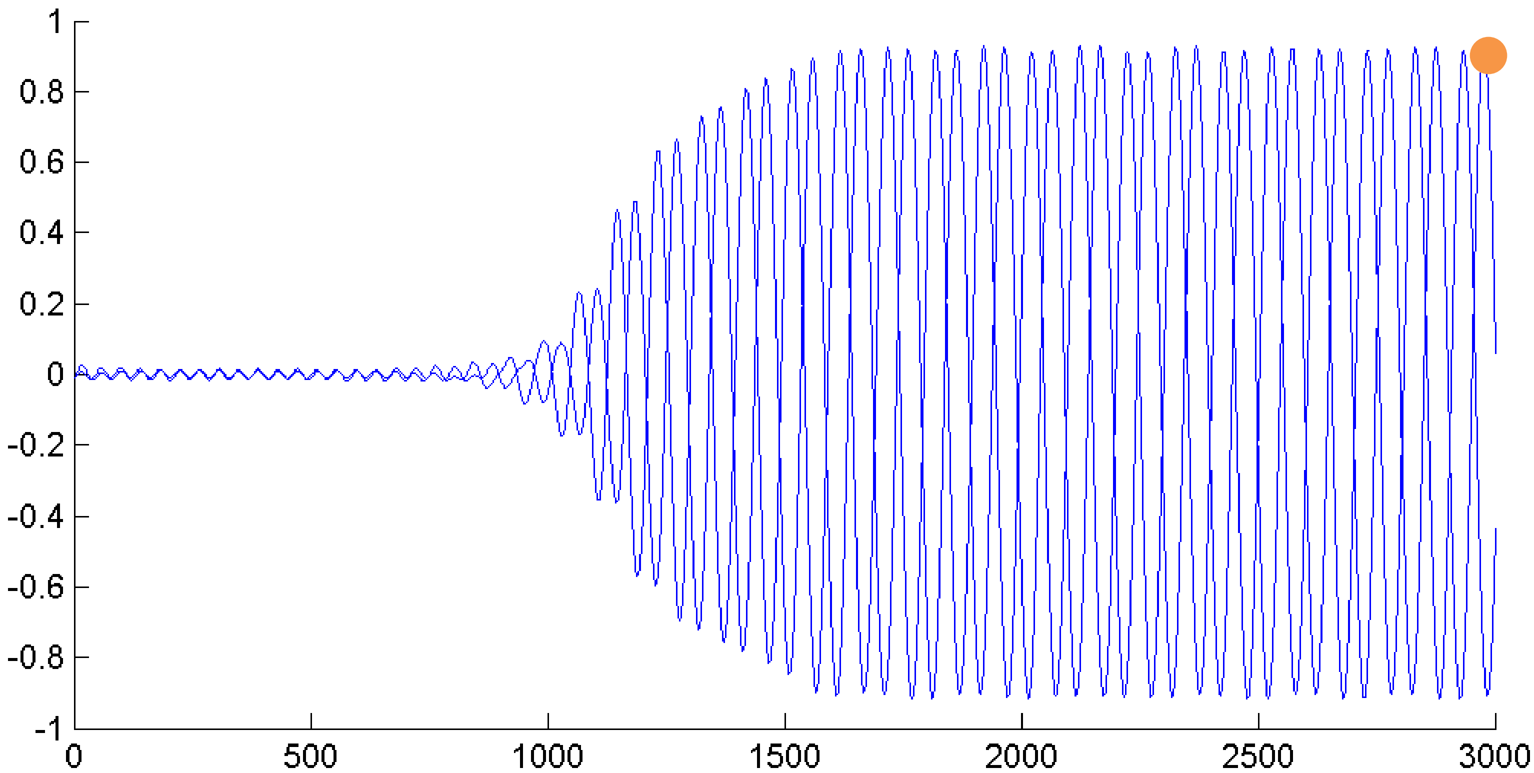
1. For a given (α, β) there exist n convergence points on the Poincaré map as determined by the Arnold Tongue diagram.

Basins of Attraction



2. Given an initial condition (x*,y*)…

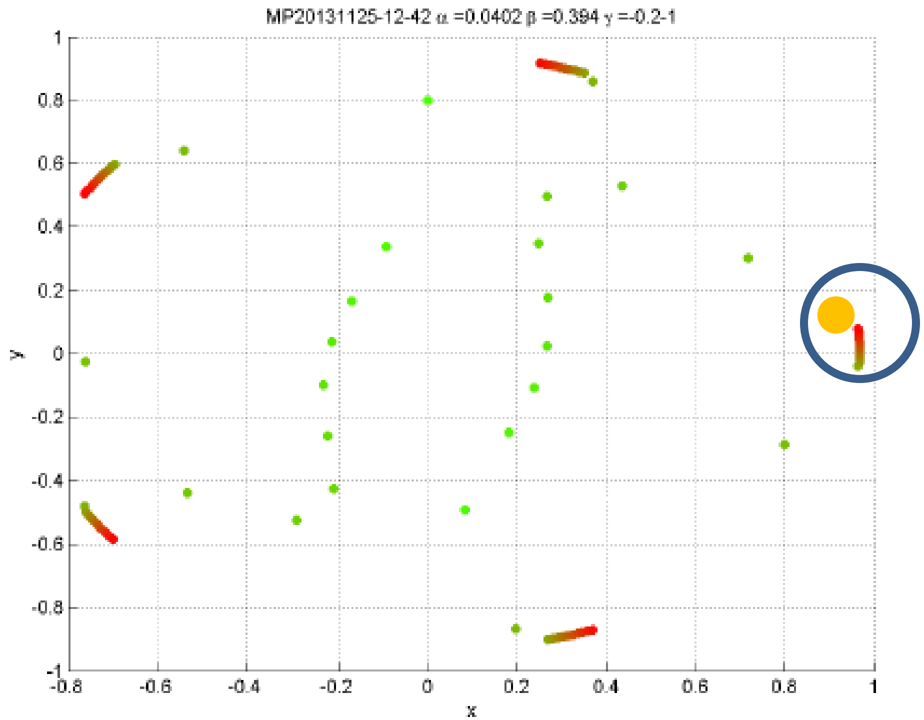
Basins of Attraction



2. Given an initial condition (x*,y*), we integrate to find $T_{(x^*,y^*)}$ for a predetermined number of simulated years.

3. We determine the (x,y) values of the last peak of simulated $T_{(x^*,y^*)}$.

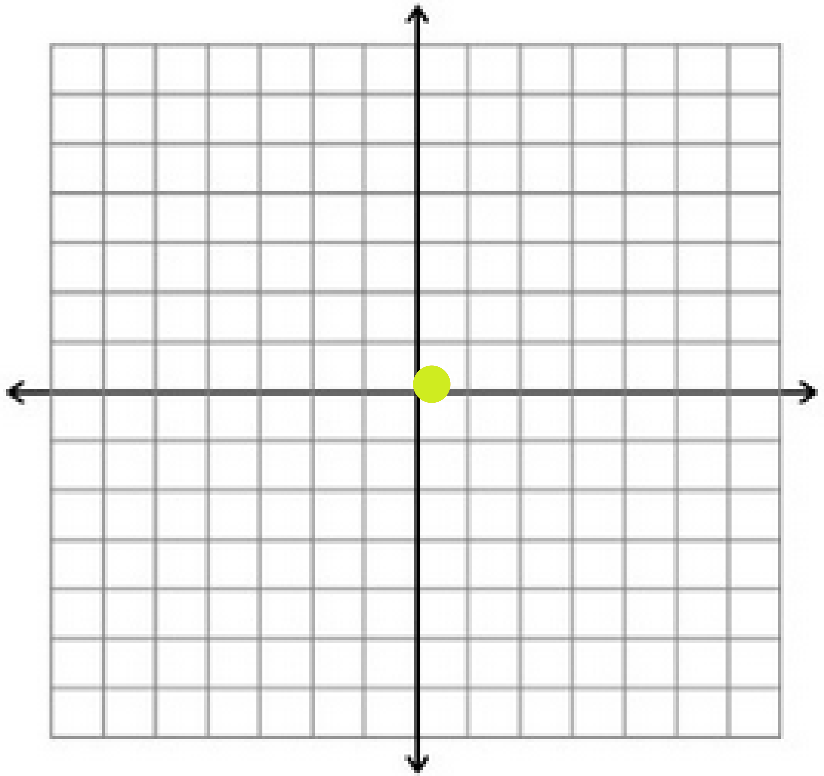
Basins of Attraction



$$p_1 = (0.04, 0.39)$$

4. We identify which convergence point on the Poincaré section the last peak of $T_{(x^*,y^*)}$ is closest to. The closeness criterion is satisfied if the Euclidian distance is less than some small threshold to guarantee closeness.

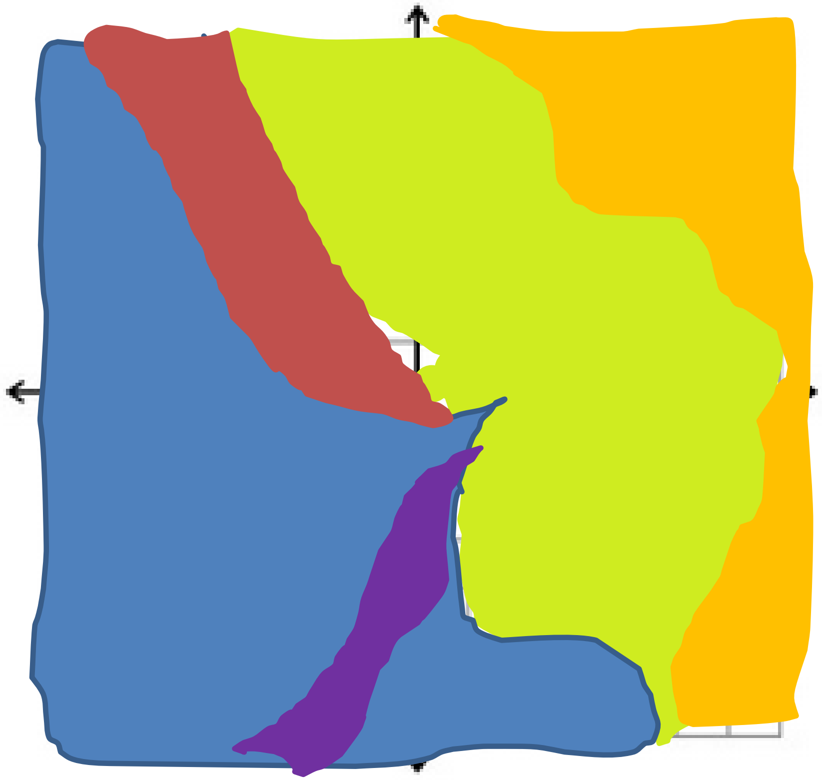
Basins of Attraction



5. Finally, (x*,y*) is assigned the color associated to the nearby convergence point.

Thus, we will complete this process for a grid of initial conditions to color code the domain based on the relative phase of $T_{(x^*,y^*)}$.

Basins of Attraction



5. Finally, (x^*,y^*) is assigned the color associated to the nearby convergence point.

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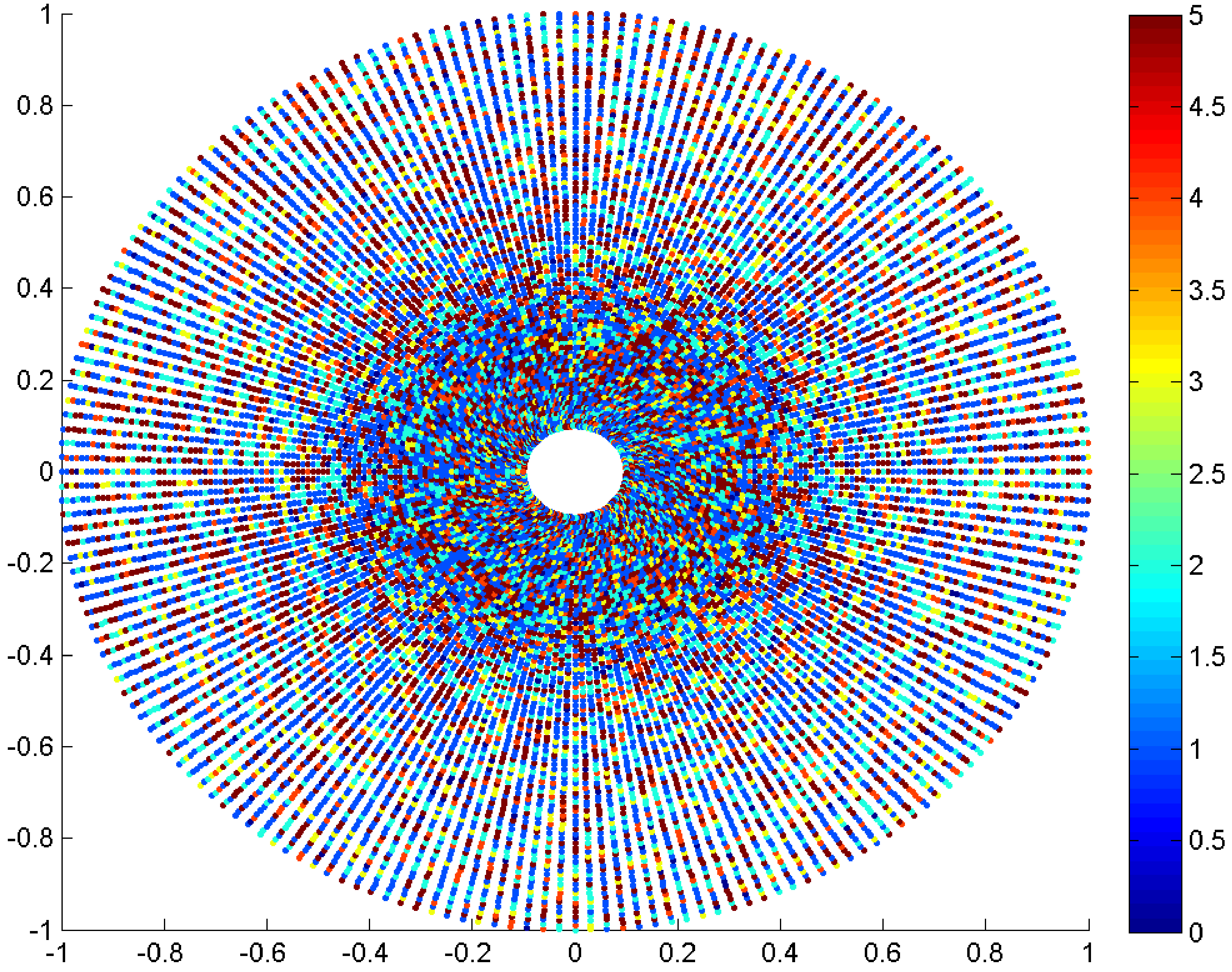
Basins of Attraction



Mathematical Conjecture

Let H be a system of equations which contains a dynamic Hopf bifurcation with constant parameters p in R^n, bifurcation parameter, γ, and initial condition, (x, y).

Conjecture. If, for a given p and time span, γ_{final} yields a system state for which multiple stable trajectories exist then the phase of $T_{(x,y)}$ after γ_{crit} is sensitive to initial conditions.

Corollary. If the planet's underlying structure is a Hopf bifurcation, then the phase of the planet's recent 100 kyr cycles has no meaningful relationship to the external forcing.

Applied Climate Conjecture

Assume, by way of contradiction, that the planet is well modeled by a dynamic Hopf bifurcation. Then:

1. We must assume there is an internal cycle on the planet that has exactly 100 kyr period. There is no evidence for such a cycle in the biosphere, cryosphere, hydrosphere, or atmosphere.

2. We must also assume that after the system underwent the Hopf bifurcation that there exists a stable trajectory which correlates with the external forcing and that T_{Earth} landed on that trajectory. This would be a cosmic coincidence with a small probability.

3. Despite the stochastic and chaotic features of the planet, we must further assume T_{Earth} has not moved from one stable trajectory to another over the last 1.2 myr.

While there is no perfect contradiction, the probability of any (much less all) of these statements being true is quite small. Thus it is reasonable to assume that the planet does not have the underlying structure of a Hopf bifurcation.

Conclusion

The post-MPT phase angle results can not be predicted for a dynamic Hopf bifurcation model. Thus, it is a matter of chance and choice of initial condition that the model results will match the $\delta^{18}O$ data analysis.

Thus, this work strongly suggests that a dynamic Hopf bifurcation will **not** successfully model the mid-Pleistocene transition.

Thanks!

Questions?