MATH3283W LECTURE NOTES: WEEK 12

4/12/2010

Power series(cont.)

Examples:

(1) $f(x) = (1+x)^{\frac{1}{2}}$

$$\begin{aligned} f'(x) &= \frac{1}{2}(1+x)^{-\frac{1}{2}} \\ f''(x) &= \frac{1}{2}\cdot(-\frac{1}{2})(1+x)^{-\frac{3}{2}} \\ f'''(x) &= \frac{1}{2}\cdot(-\frac{1}{2})\cdot(-\frac{3}{2})(1+x)^{-\frac{5}{2}} \\ f^{(4)}(x) &= -\frac{1\cdot 3\cdot 5}{2^4}(1+x)^{-\frac{7}{2}} \\ &\vdots \\ f^{(n)}(x) &= (-1)^{n+1}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{2^n}(1+x)^{-\frac{2n-1}{2}} \end{aligned}$$

So $f^{(n)}(0) = (-1)^{n+1}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{2^n}$ and

$$TS = 1 + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2^n}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{n!}x^n$$

(2) $f(x) = \sin x$ at $x = \frac{\pi}{2}$.

$f^{(n)}(x)$:	$\sin x$	$\cos x$	$-\sin x$	$-\cos x$	$\sin x$	$\cdots$
$f^{(n)}(\frac{\pi}{2})$:	1	0	-1	0	1	$\cdots$

So

$$T(\frac{\pi}{2}) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(x-\frac{\pi}{2})^{2n}$$

We have shown that the Taylor series of $\sin x$ at $x = 0$ is

$$\sin x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$$

From this example, we can see that the Taylor series of a function at different points can be very different. Moreover,

$$T_3(0) = x - \frac{x^3}{3!}, \quad T_3(\frac{\pi}{2}) = 1 - \frac{(x-\frac{\pi}{2})^2}{2!}$$

These two polynomials even have different degrees!

(3) Let $f(x) = a_0 + a_1 x + \cdots + a_n x^n$. $f(x)$ is a polynomial and

$$f(0) = a_0, f'(0) = a_1, f''(0) = 2a_2 = 2!a_2, \cdots$$

Note that $f^{(m)}(x) = 0$ if $m > n$. So

$$T(0) = a_0 + \cdots + a_n x^n$$

It means that any polynomial of x is its Taylor series.

(4) Consider $f(x) = e^{x^2}$. Since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ holds for all real x, replace x with x^2 can get

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} = 1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots$$

If we directly compute $T_n(0)$, we have

$$f(0) = 1, f'(0) = 2xe^{x^2}|_0 = 0, f''(0) = 2e^{x^2} + 4xe^{x^2}|_0 = 2, \cdots$$

and hence $T_0(0) = 1, T_1(0) = 1, T_2(0) = 1 + x^2, \cdots$, i.e. the above series is the Taylor series for e^{x^2}.

(5) Consider $P(x) = 2+x-x^3+x^5$. It is clear that $T_3(0) = 2+x-x^3$, we want to find $T_3(2)$.
We know that

$$T_3(2) = P(2) + P'(2)(x-2) + \frac{P''(2)}{2!}(x-2)^2 + \frac{P'''(2)}{3!}(x-2)^3$$

$$\begin{aligned} P(2) &= 2+2-8+32 = 28 \\ P'(2) &= 1 - 3\cdot 4 + 5 \cdot 16 = 69 \\ P''(2) &= -3 \cdot 2cdot2 + 5 \cdot 4 \cdot 8 = 148 \\ P'''(3) &= -6 + 60 \cdot 4 = 234 \end{aligned}$$

So

$$T_3(2) = 28 + 69(x-2) + 74(x-2)^2 + 39(x-2)^3$$

Differentiating and Integrating Power Series

We now show how to differentiate and integrate power series by doing the obvious way: computing term-by-term. This will allow us to find the power series for many more functions. A key result is

Theorem 0.1. *If $\sum_{k=0}^{\infty} a_k x^k$ converges on $(-c, c)$, then the power series formula by term-by-term differentiation*

$$\sum_{k=0}^{\infty} \frac{d}{dx}(a_k x^k) = \sum_{k=1}^{\infty} k a_k x^{k-1} \quad (\textit{Remark: } k \textit{ starts from } 1)$$

also converges on $(-c, c)$.

Proof. Pick $t \in (-c, c)$ and $\varepsilon > 0$ s.t. $|t| < |t| + \varepsilon < c$. Since $|t| + \varepsilon < R$ (R: the radius of convergence), $\sum |a_k(|t| + \varepsilon)^k|$ converges. Now

$$(k|t|^{k-1})^{\frac{1}{k}} = k^{\frac{1}{k}} |t|^{1-\frac{1}{k}} \overset{k \to +\infty}{\longrightarrow} 1 \cdot |t| = |t|$$

$$\text{So} \quad \exists k_0, k \geq k_0 \Rightarrow (k|t|^{k-1})^{\frac{1}{k}} < |t| + \varepsilon \; i.e. \; k|t|^{k-1} < (|t| + \varepsilon)^k$$

Hence, by CT, $\sum_{k=k_0}^{\infty} |a_k| k |t|^{k-1} = \sum_{k=k_0}^{\infty} |k a_k t^{k-1}|$ converges and so does $\sum_{k=1}^{\infty} |k a_k t^{k-1}|$. □

Corollary 0.2.
(1) *$\sum a_k x^k$ and $\sum k a_k x^{k-1}$ have the same radius of convergence R.*
(2) *If $f(x) = \sum a_k x^k$, then $f'(x) = \sum k a_k x^{k-1}$.*

Proof.
(1) From theorem 0.1, we know that $\sum k a_k x^{k-1}$ converges on $(-R, R)$. Now suppose R_1 is the radius of converges for $\sum k a_k x^{k-1}$ and $|x_1| < R_1$. Then $\sum |k a_k x_1^{k-1}|$ converges. Since $|a_k x_1^{k-1}| \leq |k a_k x_1^{k-1}|$, we know that $\sum |a_k x_1^{k-1}|$ converges, as well as $|x_1| \sum |k a_k x_1^{k-1}| = \sum |a_k x_1^k|$. So $R_1 \leq R$, which means that $R_1 = R$
(2) skip proof.

□

We know can compute all derivatives of $f(x)$ in the same way.

$$f''(x) = \sum_{k=0}^{\infty} \frac{d^2}{dx^2}(a_k x^k) = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2}$$

$$f'''(x) = \sum_{k=3}^{\infty} k(k-1)(k-2) a_k x^{k-3} \; etc$$

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Enrichment: partial sum of divergent positive series.
Just as rearrangements of conditional convergent series giving any real numbers, divergent series have similar properties for summing rearranged terms. Note that $\sum_{k=1}^{\infty} \frac{1}{n}$ diverges. We have:

Any positive rational number can be written as a finite sum of terms from $\sum_{k=1}^{\infty} \frac{1}{n}$.

Examples:

(1) $\frac{27}{37}$.
Find smallest n_1 s.t. $\frac{1}{n_1} < \frac{27}{31}$. $n_1 = 2$ because $\frac{1}{2} < \frac{27}{31}$ and $\frac{1}{1} > \frac{27}{31}$.
If we look at $\frac{27}{31} - \frac{1}{2} = \frac{54-31}{62} = \frac{23}{62}$, then $n_2 = 3$ since $\frac{1}{3} < \frac{23}{62}$.
$\frac{23}{62} - \frac{1}{3} = \frac{69-62}{186} = \frac{7}{186}$. Since $\frac{186}{7} = 26.57$, we know that $n_3 = 27$ and $\frac{7}{186} = \frac{1}{27} = \frac{189-186}{2322} = \frac{3}{2322} = \frac{1}{774}$.
So $n_4 = 774$ and we are done:

$$\frac{27}{31} = \frac{1}{2} + \frac{1}{3} + \frac{1}{27} + \frac{1}{774}$$

(2) $\frac{3}{7}$.
$n_1 = 3$ since $\frac{1}{3} < \frac{3}{7}$ and $\frac{3}{7} - \frac{1}{3} = \frac{9-7}{21} = \frac{2}{21}$.
$n_2 = 311$ since $\frac{1}{11} < \frac{2}{21}$ and $\frac{2}{21} - \frac{1}{11} = \frac{22-21}{231} = \frac{1}{231}$.
$n_3 = 231$ and $\frac{3}{7} = \frac{1}{3} + \frac{1}{11} + \frac{1}{231}$.
Check:

$$\frac{1}{3} + \frac{1}{11} + \frac{1}{231} = \frac{77+21+1}{231} = \frac{99}{231} = \frac{9}{21} = \frac{3}{7}$$

Note: the sum is unique by fact in number theory.
Exercises: Do the above for $x = \frac{2}{3}, x = \frac{7}{9}, x = \frac{28}{41}$.

(3) $\frac{10}{7} = 1 + \frac{3}{7}$.
We know that $\frac{3}{7} = \frac{1}{3} + \frac{1}{11} + \frac{1}{231}$, we can't use $3, 11$ and 231 to compute 1. We start with $\frac{1}{2} + \frac{1}{4} + \frac{1}{5} = \frac{19}{20} < 1$ which is the largest possible partial sum of the series $\sum_{k=1}^{\infty} \frac{1}{n}$ removing $\frac{1}{3}, \frac{1}{11}, \frac{1}{231}$ and less than 1. Then $1 - \frac{19}{20} = \frac{1}{20}$ and we have $1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{5} + \frac{1}{20}$.
Thus

$$\frac{10}{7} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{11} + \frac{1}{20} + \frac{1}{231}$$

Exercises: Do the above for $x = \frac{5}{3}$ and $x = \frac{25}{9}$.

(4) The above works for the odd harmonic series $\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\cdots$ and rational number $\frac{p}{q}$ with q odd. For example, if $x=\frac{7}{13}$, the largest $\frac{1}{n}$ wit h n odd and less than $\frac{7}{13}$ is $\frac{1}{3}$, so $n_1=3$ and $\frac{7}{13}-\frac{1}{3}=\frac{21-13}{39}=\frac{8}{39}$.
$\frac{1}{5}<\frac{8}{39}$ and $n_2=5$, $\frac{8}{39}-\frac{1}{5}=\frac{40-39}{195}=\frac{1}{195}$.
S0 $n_3=195$ and we have $\frac{7}{13}=\frac{1}{3}+\frac{1}{5}+\frac{1}{195}$.
For full harmonic series, we have $\frac{7}{13}=\frac{1}{2}+\frac{1}{26}$.

Integration of power series

Consider a PS $\sum_{k=0}^{\infty}a_kx^k$ and the PS formed by term-by-term integration:

$$\sum_{k=0}^{\infty}\int a_kx^k dx=\sum_{k=0}^{\infty}\frac{a_kx^{k+1}}{k+1}$$

If R is the radius of converges for this new series, then since $\frac{d}{dx}(\frac{a_kx^{k+1}}{k+1})=a_kx^k$, the original series also has radius of convergence R as before. If $f(x)=\sum_{k=0}^{\infty}a_kx^k$, then

$$\begin{aligned}\int f(x)dx &= \sum_{k=0}^{\infty}\int a_kx^k dx=\sum_{k=0}^{\infty}\frac{a_kx^{k+1}}{k+1}\\ &= C+a_0x+a_1x^2+a_2\frac{x^3}{3}+\cdots,\ \text{C: constant of integration}\end{aligned}$$

Examples:

(1) We have

$$\sin x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots$$

Then

$$\begin{aligned}\cos x &= \frac{d}{dx}(\sin x)=\sum_{n=0}^{\infty}\frac{d}{dx}((-1)^n\frac{x^{2n+1}}{(2n+1)!})\\ &= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}(2n+1)}{(2n+1)!}=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots\end{aligned}$$

(2) Consider

$$\ln(x)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(x-1)^n, 0<x\leq 2$$

Replace $x-1$ by w. Since $0<x\leq 2, -1<x-1\leq 1$. So $-1<w\leq 1$ and we get $\ln(1+w)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}w^n$ with

$R=1$ as before, but now it is a PS with w at $w=0$. We can form a PS for

$$\ln(\frac{1+x}{1-x}) = \ln(1+x) - \ln(1-x) = \sum_{n=1}^{\infty}(\frac{(-1)^{n+1}}{n} - a_n)x^n$$

where $\sum_{n=1}^{\infty} a_n x^n = \ln(1-x)$. But

$$\ln(1-x) = \int \frac{-1}{1-x}dx = (-1)\sum_{n=0}^{\infty}\int x^n dx = (-1)\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty} -\frac{x^n}{n}\ (C=0)$$

So $\frac{(-1)^{n+1}}{n} - \frac{1}{n} = \begin{cases} \frac{2}{n}, & n = 2k+1 : odd \\ 0, & n = 2k : even \end{cases}$ and $R=1$.

$$\ln(\frac{1+x}{1-x}) = \sum_{k=0}^{\infty}\frac{2}{2k+1}x^{2k+1} = 2\sum_{k=0}^{\infty}\frac{x^{2k+1}}{2k+1}$$

This allows us to estimate $\ln w$ for any positive w. If we solve $\frac{1+x}{1-x} = w$ for $0 < x < 1$,

$$1 + x = w(1-x) = w - wx,\ (1+w)x = w-1,\ x = \frac{w-1}{w+1}$$

$w = 9, x = .8$,

$$\ln 9 \approx 2[\frac{.8}{1} + \frac{.8^3}{3} + \frac{.8^5}{5}] \approx 2 \cdot 1.0362 = 2.0724$$

(A modest estimate is 2.1972. Adding $\frac{.8^7}{7}$ gives 2.1023)
$w = 2, x\frac{1}{3}$,

$$\ln 2 \approx 2 \times [\frac{1}{3} + \frac{(\frac{1}{3})^3}{3} + \frac{(\frac{1}{3})^5}{5}] = 2 \times .3465 = .6930$$

an excellent estimate ($\ln 2 \approx .69315$).

(3) We have $e^{x^2} = \sum_{n=0}^{\infty}\frac{x^{2n}}{n!}$ (start $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$, and replace x by x^2). Suppose we want to estimate $\int_0^1 e^{x^2}dx$. Now

$$\int e^{x^2}dx = \sum_{n=0}^{\infty}\int\frac{x^{2n}}{n!}dx = \sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)\cdot n!} = x + \frac{x^3}{3\cdot 1!} + \frac{x^5}{5\cdot 2!} + \frac{x^7}{7\cdot 3!} + \cdots$$

So $\int_0^1 e^{x^2}dx \approx x + \frac{x^3}{3} + \frac{x^5}{10} + \frac{x^7}{42}|_0^1 = 1 + \frac{1}{3} + \frac{1}{10} + \frac{1}{42} = 1.4571$

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(4) The geometric series together with differentiation/integration of power sries gives us lots of new PS. Since we know

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n}$$

We can get

$$\begin{aligned}\arctan x &= \int \frac{1}{1+x^2}dx = \sum_{n=0}^{\infty}\int(-1)^n x^{2n}dx = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1} + C \\ &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + C \\ &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \quad (C = \arctan 0 = 0)\end{aligned}$$

This PS converges for $R = 1$.
At $x = -1$, we get $\sum_{n=0}^{\infty}(-1)^n\frac{(-1)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{2n+1}$ which converges by alternating series test.
At $x = 1$, we get $\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$ which also converges .

Theorem 0.3 (Abel Theorem)**.** *(see textbook)*
If $f(x) = \sum_{n=0}^{\infty} a_k x^k$ converges for $|x| < 1$ and $\sum_{n=0}^{\infty} a_k$ converges, then $f(1) = \sum_{n=0}^{\infty} a_k$ when $f(1)$ is continuous at $x = 1$.

Thus $\frac{\pi}{4} = \arctan 1 = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$
(Note: this is amazing because an irrational number $\frac{\pi}{4}$ can be expressed as sum of rational numbers. But this PS is not useful because it converges too slowly.)

Example: Find power series for $f(x) = \frac{1}{(1+x^2)^2}$.
We can multiply PS of $\frac{1}{1+x^2}$ to get it. An alternate way is to use the following formula:

$$\frac{d}{dx}\left(\frac{1}{1+x^2}\right) = (-1)\frac{1}{(1+x^2)^2}\cdot 2x = \frac{-2x}{(1+x^2)^2}$$

Since $\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n}$,

$$\frac{d}{dx}\left(\frac{1}{1+x^2}\right) = \sum_{n=1}^{\infty}(-1)^n 2n\cdot x^{2n-1} = 2\sum_{m=0}^{\infty}(-1)^{m+1}(m+1)\cdot x^{2m+1} \quad (m = n-1)$$

$$\text{So } \frac{-2x}{(1+x^2)^2} = 2\sum_{n=0}^{\infty}(-1)^{n+1}(n+1)\cdot x^{2n+1}$$

$$\frac{x}{(1+x^2)^2} = \sum_{n=0}^{\infty}(-1)^{n}(n+1)\cdot x^{2n+1}$$

$$\frac{1}{(1+x^2)^2} = \sum_{n=0}^{\infty}(-1)^{n}(n+1)\cdot x^{2n} = 1-2x^2+3x^4-4x^6+\cdots$$

Test: when $x = .17$, $\frac{1}{(1+x^2)^2} \approx .9446$ and the first 4 terms $\approx .9536$. So the error $\approx .009$.

Counting sets: levels of infinity

Recall: $A \neq \emptyset$ is finite if $\exists n \in \mathbb{N}$ such that A has precisely n elements:

$$A = \{a_i | 1 \le i \le n\} \quad \& \quad i \neq j \Rightarrow a_i \neq a_j$$

Another way: if $\mathbb{N}_n = \{1, 2, \cdots, n\}$, then there is an 1-to-1 and onto function $f : \mathbb{N}_n \to A, f(i) = a_i$.
(Onto means that the set Range$f = \{y | y = f(x) \text{ for some } x\}$ is equal to A.) 1-to-1 and onto functions are also called bijections.

Definition 0.4. *$f : C \to D$ is 1-to-1 if*

$$\forall c_1, c_2 \in C, c_1 \neq c_2 \Rightarrow f(c_1) \neq f(c_2).$$

$f : C \to D$ is onto if

$$\forall d \in D, \exists c \in C \Rightarrow f(c) = d$$

This is the corresponding definition of finite:
$A \neq \emptyset$ is finite if $\exists n \in \mathbb{N}$ and a bijection $f : \mathbb{N}_n \to A$.

Examples:

(1) $f_j, g_j : \mathbb{N} \to \mathbb{N}, f_j(n) = n + j, g_j(n) = jn$ where $j \in \mathbb{N}$ is fixed. (For instance, $f_2(n) = n + 2, g_2(n) = 2n$.) Both are 1-to-1, but not onto:
If $j \neq 1$, suppose $f_j(n_1) = f_j(n_2)$, so $n_1+j = n_2+j$ and $n_1 = n_2$. This is the contrapositive of $n_1 \neq n_2 \Rightarrow f_j(n_1) \neq f_j(n_2)$. So f_j is 1-to-1.
Since $jn_1 = jn_2 \to n_1 = n_2$, g_j is also 1-1.
If $j \neq 1, f_j(\mathbb{N}) = \{n > j | n \in \mathbb{N}\} = \{1 + j, 2 + j, 3 + j, \cdots\}$ and $1, 2, \cdots, j \notin f_j(\mathbb{N})$. So f_j is not onto. (Note: in this case, it is true even if $j = 1$.)
$g_j(\mathbb{N}) = \{jn | n \in \mathbb{N}\} = \{j, 2j, 3j, \cdots\}$ and $1, 2, \cdots, j - 1 \notin$

$g_j(\mathbb{N})$. So g_j is not onto.
If $j = 1$, $g_j(n) = 1 \cdot n = n$ and g_1 is onto (so a bijection).

(2) Suppose A is a finite set and $f : \mathbb{N}_n \to A, f(j) = a_j$ a bijection. Then $g : \mathbb{N}_n \to A, g(j) = a_{n+1-j}$ is also a bijection.

We can use these ideas to refine our notion of infinitity (previous definition: not finite.)

Definition 0.5. *A is countably infinite if* $\exists f : \mathbb{N} \to A$ *such that* f *is a bijection. Setting* $f(i) = a_i$, *we can express* $A = \{a_i | i \in \mathbb{N}\}$ *and* $i \neq j \Rightarrow a_i \neq a_j$

Example: The set O of odd positive integers and set E of even positive integers are countably infinite. The functions $f : \mathbb{N} \to E, f(n) = 2n$ and $g : \mathbb{N} \to O, g(n) = 2n - 1$ are bijections (Exercise).