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Power Series

Consider a series $\sum_{n=0}^{\infty} a_n x^n$ (where (a_n) is a sequence & x is a variable). We ask ourselves a natural question: For what values of x does this series converge? Turns out we have three possible answers:

① Converges for $x=0$ only

② Converges for $x\in(-r,r)$ (or $[-r,r)$, or $(-r,r]$, or $[-r,r]$) for some $r\in(0,\infty)$

③ Converges for all x.

We define the radius of convergence to be the largest number R (possibly infinite) s.t. the series converges $\forall x\in(-R,R)$. In particular in case ① $R=0$, in case ② $R=r$ & in case ③ $R=\infty$. The interval of convergence is the interval $(-R,R)$ (where R is the radius of convergence) with possible also $\{-R\}$ or $\{R\}$ (if the series converges at $x=-R$ or $x=R$). So a series (with finite radius of convergence) will have an interval of convergence of exactly one of the following forms:

$(-R,R)$, $[-R,R)$, $[-R,R]$ or $(-R,R]$.

Examples find R.O.C. & I.O.C. of the following

(5) $\sum_{n=1}^{\infty} 5^n \cdot n x^n$

(6) $\sum_{n=1}^{\infty} \frac{x^n}{n^p}$ (for fixed $p>1$)

Ratio test for ROC
End points by inspection

(7) $\sum_{n=1}^{\infty} (3^{\frac{1}{n}}+4^{\frac{1}{n}})^n x^n$

For ROC we use Root test:

$$\lim_{n\to\infty} |(3^{\frac{1}{n}}+4^{\frac{1}{n}})^n x^n|^{\frac{1}{n}} = \lim_{n\to\infty}(3^{\frac{1}{n}}+4^{\frac{1}{n}})|x| = (1+1)|x| = 2|x|.$$ So

Series converges if $\lim_{n\to\infty} |(3^{\frac{1}{n}}+4^{\frac{1}{n}})^n x^n|^{\frac{1}{n}} = 2|x| < 1$ i.e, if $x\in(-\frac{1}{2},\frac{1}{2})$

Series diverges if $\lim_{n\to\infty} |(3^{\frac{1}{n}}+4^{\frac{1}{n}})^n x^n|^{\frac{1}{n}} = 2|x| > 1$. Check the

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endpoints: at $x=\frac{1}{2}$, we have

$$\sum (3^{\frac{1}{n}}+4^{\frac{1}{n}})^n\left(\frac{1}{2}\right)^n = \sum \left[\frac{1}{2}(3^{\frac{1}{n}}+4^{\frac{1}{n}})\right]^n \geq \sum 1$$

diverges by comparison

at $x=-\frac{1}{2}$ $\sum (3^{\frac{1}{n}}+4^{\frac{1}{n}})^n\left(\frac{-1}{2}\right)^n$ diverges since $\lim_{n\to\infty} (3^{\frac{1}{n}}+4^{\frac{1}{n}})^n\left(-\frac{1}{2}\right)^n$ DNE (convergent series must have limit of summands equal to 0).

(8) $\sum_{n=0}^{\infty} x^n$ (which is ~~ex~~ exactly the function $\frac{1}{1-x}$ on the interval of convergence)

We know series converges iff $|x|<1$ (geometric series)

$\sum_{n=0}^{\infty}\left(\frac{x^n}{n!}+x^n\right)$ - rewrite as $\sum_{n=0}^{\infty}\left(1+\frac{1}{n!}\right)x^n$ & use ratio test. Note has IOC same as $\sum_{n=0}^{\infty} x^n$, even though $\sum \frac{x^n}{n!}$ converges everywhere

Fact: Let $\sum a_n x^n$ & $\sum b_n x^n$ be power series with radii of convergence R & S. Then the power series $\sum (a_n+b_n)x^n$ has radius of convergence $\geq \min(R,S)$, & in particular, if $R\neq S$ then the radius of convergence of $\sum (a_n+b_n)x^n$ is equal to $\min(R,S)$.

Exercise: Find two power series $\sum a_n x^n$ & $\sum b_n x^n$ with the same radius of convergence, but $\sum (a_n+b_n)x^n$ has strictly greater ROC.

(9) $\sum_{n=0}^{\infty} x^{n^2}$ — rewrite as a power series — $\sum_{m=0}^{\infty} a_m x^m$ where $a_m = \begin{cases} 1 & \text{if } \sqrt{m}\in\mathbb{Z} \\ 0 & \text{otherwise}\end{cases}$. Compare $\sum |a_m x^m|$ to $\sum |x|^m$ — thus ROC is at least 1 since $|a_m x^m| \leq |x|^m$ & $\sum |x|^m$ converges for all $x\in(-1,1)$. To check that ROC is infact equal to 1, we show that $\sum_{n=0}^{\infty} x^{n^2}$ diverges for $x=1$ (or $x=-1$)

NOTE: BOTH ROOT & RATIO TEST FAIL.

(10) Consider the series: $1+2x+x^2+2x^3+x^4+2x^5+\cdots$

We write as $\sum_{n=0}^{\infty} x^{2n} + \sum_{n=0}^{\infty} 2x^{2n+1}$ then we have

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$\sum_{n=0}^{\infty} x^{2n} = \frac{1}{1-x^2}$ & $\sum_{n=0}^{\infty} 2x^{2n+1} = \sum_{0}^{\infty} 2x \cdot (x)^{2n} = 2x \cdot \left(\sum_{0}^{\infty} x^{2n}\right) = \frac{2x}{1-x^2}$

so $1+2x+x^2+2x^3+\cdots = \left(\sum x^{2n}\right) + \left(\sum 2x^{2n+1}\right) = \frac{1}{1-x^2} + \frac{2x}{1-x^2} = \frac{1+2x}{1-x^2}$

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Sums of Power series & associated radii of convergence

We know: If $\sum a_n x^n$ & $\sum b_n x^n$ are power series with radii of convergence R & S, respectively. Then, if T is the radius of convergence of $\sum (a_n+b_n)x^n$, then

$$T \geq \min(R,S);$$

moreover, if $R \neq S$, $T = \min(R,S)$.

The case when $R=S$ we have a more complicated situation:

Ex 1) $a_n = \frac{1}{n}$ $\quad b_n \equiv 1 \ \forall n$ then

$\sum a_n x^n = \sum \frac{x^n}{n}$ has ROC $= 1$ &

$\sum b_n x^n = \sum x^n$ has ROC$=1$.

Now $\sum (a_n+b_n)x^n = \sum (\frac{1}{n}+1)x^n$ Also has ROC

Ex 2) $a_n = \frac{1}{n}$ $\quad b_n = \frac{n-2^n}{(n2^n)+1}$

$\sum a_n x^n$ has ROC $= 1$

$\sum b_n x^n$ has ROC$= 1$ ← Check!

but $\sum (a_n+b_n)x^n$ has ROC$=2$ ← Check!

Multiplying Series

Given power series $\sum_{n=0}^{\infty} a_n x^n$ $\sum_{n=0}^{\infty} b_n x^n$, we would like to define a product $(\sum a_n x^n)\cdot(\sum b_n x^n)$ which is again a power series. In other words, given series above we need to define $\langle c_n \rangle$ such that

$$(\sum a_n x^n)(\sum b_n x^n) = \sum c_n x^n \quad \forall x \text{ s.t. } \sum a_n x^n \text{ \& } \sum b_n x^n \text{ both converge.}$$

How to do this? We think of Power series as "limits of polynomials" i.e, $\sum_{n=0}^{\infty} a_n x^n = \lim_{N\to\infty} (\sum_{n=0}^{N} a_n x^n)$. We know how to multiply polynomials: Namely, ~~$(\sum_{n=0}^{N} a_n x^n)(\sum_{n=0}^{N} b_n x^n) = a_0b_0 + (a_1b_0 + a_0b_1)x + \cdots$~~

$$(\sum_{n=0}^{N} a_n x^n)(\sum_{n=0}^{N} b_n x^n) = a_0b_0 + (a_1b_0 + a_0b_1)x + \cdots + (a_kb_0 + a_{k-1}b_1 + \cdots + a_1b_{k-1} + a_0b_k)x^k + \cdots + a_Nb_N x^{2N}$$

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In particular, for polynomials $\left(\sum_{n=0}^{N} a_n x^n\right)$ $\left(\sum_{m=0}^{M} b_m x^m\right)$, we have

$$\left(\sum_{n=0}^{N} a_n x^n\right)\left(\sum_{m=0}^{M} b_m x^m\right) = \left(\sum_{k=0}^{N+M} c_k x^k\right) \quad (*)$$

where $c_k = \sum_{j+i=k} a_j b_i$ (this sum is taken over all pairs (j,i) s.t. $j \in \{0,1,\ldots,N\}$, $i \in \{0,1,\ldots,M\}$ & $j+i=k$.

Hence to get the product of power series we "take" the limit of $(*)$ i.e.

$$\lim_{M,N\to\infty} \left(\sum_{n=0}^{N} a_n x^n\right)\left(\sum_{m=0}^{M} b_m x^m\right) = \lim_{M,N\to\infty} \left(\sum_{k=0}^{M+N} c_k x^k\right)$$

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{m=0}^{\infty} b_m x^m\right) = \sum_{k=0}^{\infty} c_k x^k$$

Where $c_k = \sum_{j+i=k} a_j b_i = \sum_{j=0}^{k} a_j b_{k-j}$

Given a function $f(x)$ which can be differentiated infinitely often at $x=0$, can we find a power series for $f(x)$?
The method is to compute Taylor polynomials and then form Taylor series.

Taylor Polynomials Idea: form polynomials $T_n(x)$ of degree n which have the same derivatives as $f(x)$ at 0. (By convention, the 0^{th} derivative of $f(x)$ is $f(x)$.)

$T_0(x) = f(0)$, a constant function. $T_1(x) = f(0) + f'(0)x$, the tangent line to $f(x)$ at 0.

$T_2(x) = T_1(x) + ax^2$ for some a, such that $T_2''(0) = f''(0)$. But $T_2''(0) = 2a$, so we must have $a = \frac{1}{2}f''(0)$ and thus $T_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2$.

Repeating this process, we find $T_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$. If we let this be the n^{th} partial sum of a series, we obtain the Taylor series $T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$. (by def., $0! = 1$)

Two natural questions: where does $T(x)$ converge? Does $T(x) = f(x)$?

Examples ① $f(x) = e^x$, so $\forall n \geq 1$, $f^{(n)}(x) = e^x$ and $f^{(n)}(0) = e^0 = 1$. Therefore $T(x) = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$, which we've shown (by the ratio test) converges for all $x \in \mathbb{R}$.

(It is true – not proved here – that $e^x = T(x)$ for all $x \in \mathbb{R}$.)

② $f(x) = \sin x$.

n	0	1	2	3	4
$f^{(n)}(x)$	$\sin x$	$\cos x$	$-\sin x$	$-\cos x$	$\sin x$
$f^{(n)}(0)$	0	1	0	−1	0

repeats w/ period 4

$$\Rightarrow T(x) = 0 + \frac{x}{1!} + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!},$$ an "alternating odd series".

Does $T(x)$ converge? We use the ratio test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{|x|^{2n+1}} = \frac{|x|^2}{(2n+2)(2n+3)} \longrightarrow 0 \quad \forall x \in \mathbb{R},$$ so $T(x)$ converges $\forall x \in \mathbb{R}$.

(Again, it is a fact that $\sin(x) = T(x)$ for all $x \in \mathbb{R}$.)

③ $f(x) = \frac{1}{1-x} = (1-x)^{-1}$

n	0	1	2	3
$f^{(n)}(x)$	$(1-x)^{-1}$	$(1-x)^{-2}$	$2(1-x)^{-3}$	$3!(1-x)^{-4}$
$f^{(n)}(0)$	1	1	2	$3!$

$$\Rightarrow T(x) = 1 + x + \frac{2x^2}{2!} + \frac{6x^3}{3!} + \cdots = \sum_{n=0}^{\infty} x^n,$$ the geometric series, which converges for $|x| < 1$.

In fact, we already knew $T(x) = f(x)$ for $|x| < 1$. This was a special case of the following general fact: IF a function $f(x)$ has a power series expansion, that power series must be the Taylor series.

④ If $a \in \mathbb{R}$ is some constant, we can write $\frac{1}{a+x} = \frac{1}{a}\left(\frac{1}{1-\left(\frac{-x}{a}\right)}\right)$ (check the algebra), and find a power series (hence Taylor series) for $\frac{1}{a+x}$: it is $\frac{1}{a}\left(1 - \frac{x}{a} + \frac{x^2}{a^2} \cdots\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{a^{n+1}} x^n$, which converges when $\left|\frac{x}{a}\right| < 1$, i.e. $|x| < |a|$.

(4/9, cont.) <u>Power (Taylor) series based at $x = a$</u> (a not necessarily 0) [NBS]

In this case, we have a function $f(x)$ which can be differentiated infinitely often at $x = a$. Our Taylor polynomials will now be polynomials $T_n(x)$ in powers of $(x-a)$, with the same derivatives as $f(x)$ at a.

$$\left.\begin{aligned} T_0(x) &= f(a) \\ T_1(x) &= f(a) + f'(a)(x-a) \quad \text{(tangent line again)} \\ &\ \vdots \\ T_n(x) &= \sum_{j=0}^{n} \frac{f^{(j)}(a)}{j!}(x-a)^j \end{aligned}\right\} \Rightarrow \text{Taylor series based at } x=a \quad T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

(now we evaluate $f^{(n)}$ at a, rather than 0)

<u>Example</u> $f(x) = \ln(x)$. $\ln(0)$ is undefined, so it makes no sense to talk about a Taylor series for $\ln(x)$ at $x = 0$. Let's try $x = 1$ instead (so we evaluate derivatives of $\ln(x)$ at $x = 1$):

n	0	1	2	3		
$f^{(n)}(x)$	$\ln(x)$	$\frac{1}{x}$	$\frac{-1}{x^2}$	$\frac{(-1)(-2)}{x^3}$	$\cdots$	$((-1)^{n+1}(n-1)!)/x^n$
$f^{(n)}(1)$	0	1	-1	$2!$	$\cdots$	$(-1)^{n+1}(n-1)!$

Thus $T(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n = (x-1) - \frac{(x-1)}{2} + \frac{(x-1)}{3} \cdots$

Where does this series converge? Use the ratio test: $\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x-1|^{n+1}}{n+1} \cdot \frac{n}{|x-1|^n} = \frac{|x-1|}{\left(\frac{n+1}{n}\right)} \to |x-1|$,

so $T(x)$ converges when $|x-1| < 1$, i.e. $0 < x < 2$.

(What about the endpoints? If $x = 0$, $T(x) = -\sum_{n=1}^{\infty} \frac{1}{n}$, which clearly diverges. If $x = 2$, $T(x)$ is the alternating harmonic series, which converges... to $\ln(2)$.)