Absolute and conditional convergence

We now consider series whose terms can be positive or negative.

Def. $\sum_{n=1}^{\infty} a_n$ converges absolutely if $\sum_{n=1}^{\infty} |a_n|$ converges.

First theorem: If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} a_n$ converges. (So absolute convergence is a stronger notion.)

Proof. Note that $a_n = (a_n + |a_n|) - |a_n|$, and that $a_n + |a_n| = \begin{cases} 0 & \text{if } a_n \leq 0; \\ 2a_n = 2|a_n| & \text{if } a_n \geq 0. \end{cases}$

By hypothesis $\sum |a_n|$ converges;

moreover $\sum (a_n + |a_n|)$ converges by the comparison test, as we just showed $0 \leq a_n + |a_n| \leq 2|a_n| \; \forall n$.

But then $\sum a_n$, as a difference of convergent series, converges. ▨

Examples $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}$ converges absolutely, since $\left| \frac{(-1)^{n+1}}{n^3} \right| = \frac{1}{n^3}$, and $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges (integral test).

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$ does not converge absolutely (integral test again). However, the following theorem shows that it does converge.

Theorem (Alternating Series Test) Let $\langle a_n \rangle$ be a strictly decreasing sequence (so $a_n > a_{n+1} \; \forall n$) with positive terms, such that $a_n \to 0$.

Then the "alternating series" $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges.

Sketch of proof. It suffices to show that the even partial sums s_{2n} and odd partial sums s_{2n+1} both converge (to the same limit); what is more, since $s_{2(n+1)} = s_{2n+1} + a_{2(n+1)}$ and $a_{2(n+1)} \to 0$, we need only look at the ~~even~~ odd partial sums.

But $\langle s_{2n+1} \rangle$ is decreasing (since
$$\begin{aligned} s_{2n+3} &= s_{2n+1} - a_{2n+2} + a_{2n+3} \\ &= \Big[\underbrace{(a_1 - a_2)}_{>0} + \cdots + \underbrace{(a_{2n-1} - a_{2n})}_{>0} + \underbrace{a_{2n+1}}_{>0}\Big] - \underbrace{(a_{2n+2} - a_{2n+3})}_{>0} \\ &< s_{2n+1}; \end{aligned}$$
note that both s_{2n+1}, s_{2n+3} are positive)

and bounded below by zero, so we're done by the Monotone Convergence Theorem. ▨

"Famous" example: the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ converges by the above theorem, since $a_n = \frac{1}{n}$ satisfies all 3 conditions. (What's the limit?)

Note We can actually estimate the rate of convergence (see text): $\boxed{\forall n \geq 1, \; |s_n - L| < a_{n+1}}$. So, for example, if $n = 5$, $\left| 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - L \right| = \left| \frac{47}{60} - L \right| < \frac{1}{6}$. (Eventually, we'll identify L.)

<u>Def.</u> A series is <u>conditionally convergent</u> if $\sum_{n=1}^{\infty} a_n$ converges but <u>not</u> $\sum_{n=1}^{\infty} |a_n|$.

<u>Remark.</u> Any conditionally convergent series must have infinitely many negative terms <u>and</u> infinitely many positive terms (why?)

It is a hard theorem that if $\sum_{n=1}^{\infty} a_n$ converges conditionally and $r \in \mathbb{R}$ is a real number, there is some rearrangement of the terms of $\sum a_n$ such that the resulting series converges to r. (Proved by Riemann in 1867)

(3/31) More on <u>rearrangements</u> (enrichment topic)

Let $\sum_{n=1}^{\infty} a_n$ be a <u>conditionally</u> convergent series. Define $\mathbb{N}_+ = \{ n \in \mathbb{N} \mid a_n \geq 0 \}$ and $\mathbb{N}_- = \{ n \in \mathbb{N} \mid a_n < 0 \}$, so $\mathbb{N} = \mathbb{N}_+ \cup \mathbb{N}_-$. Form the "positive series" $\sum_{n=1}^{\infty} p_n$ where $p_n = \begin{cases} a_n, & \text{if } n \in \mathbb{N}_+; \\ 0 & \text{if } n \in \mathbb{N}_- \end{cases}$ as well as the "negative series" $\sum_{n=1}^{\infty} q_n$ where $q_n = \begin{cases} 0, & \text{if } n \in \mathbb{N}_+; \\ a_n & \text{if } n \in \mathbb{N}_- . \end{cases}$

<u>Lemma.</u> Both $\sum p_n$ and $\sum q_n$ diverge. (Here "Σ" means $\sum_{n=1}^{\infty}$.)

<u>Proof.</u> Note that $\forall n \geq 1$, $a_n = p_n + q_n$. So $p_n = a_n - q_n$ and $q_n = a_n - p_n$; therefore one of $\sum p_n$, $\sum q_n$ converges if and only if the other does. Now suppose, for contradiction, that $\sum p_n$ (and hence $\sum q_n$) converges. Then $-\sum q_n = \sum(-q_n)$ converges; hence, so does $\sum p_n - \sum q_n = \sum p_n + (-q_n)$. But $\forall n \geq 1$, $p_n + (-q_n)$ is nothing other than $|a_n|$; so $\sum |a_n|$ converges, contradicting the assertion that $\sum a_n$ only converged conditionally. ▨

<u>Example</u> Consider the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$, which converges conditionally. We find a rearrangement of its terms which sums to 2.

<u>strategy</u> Add consecutive positive (odd) terms until the sum exceeds 2, then subtract negative (even) terms until the sum is below 2; repeat. Make sure to use every term once. This works because the positive terms' sum is divergent to ∞, and the negative terms' divergent to $-\infty$, by the Lemma above.

$$1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{15} \approx 2.02 > 2, \quad \text{but} \quad \left(1 + \cdots + \frac{1}{15}\right) - \frac{1}{2} < 2.$$

$$\left[\left(1 + \cdots + \frac{1}{15}\right) - \frac{1}{2}\right] + \frac{1}{17} + \cdots + \frac{1}{41} \approx 2.004 > 2, \quad \text{but if we subtract } \frac{1}{4}, \text{ it's } < 2.$$

Keep doing this; the result will be a rearrangement with a sum of 2.

<u>Note</u> The alternating harmonic series, in its original order, converges to <u>$\ln(2)$</u>; eventually, we'll prove this.

(3/31, cont.) [NBS]

Final comments about the Alternating Series Test: all three hypotheses (① $\langle a_n \rangle$ strictly decreasing, ② $a_n > 0$, ③ $a_n \to 0$) are necessary for the conclusion. Here are two series whose terms $\langle a_n \rangle$ satisfy ②,③ but not ①:

$$\sum_{n=1}^{\infty} (-1)^{n+1} \cdot a_n = 1 - \frac{1}{2^3} + \frac{1}{3^2} - \frac{1}{4^3} + \frac{1}{5^2} \cdots$$ <u>converges</u>, but

$$\sum_{n=1}^{\infty} (-1)^{n+1} \cdot a_n = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{8} + \frac{1}{4} - \frac{1}{16} \cdots$$ <u>diverges</u>.

Recommended <u>exercises</u>: 6.12, 6.14, 6.15, 6.16 in Section 4.6.

(4/2) <u>Power Series</u>

Given any sequence $\langle a_n \rangle$, we can form the series $\sum_{n=1}^{\infty} a_n$ and test for convergence. Write $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_n \cdot 1 = \sum_{n=1}^{\infty} a_n \cdot (1)^n$, a power series at $x = 1$. Question: if we replace 1 by any $x \in \mathbb{R}$, does the resulting series $\sum_{n=1}^{\infty} a_n x^n$ converge? This is a <u>power series in x</u>, and we want to find $\{ x \in \mathbb{R} \mid \sum_{n=1}^{\infty} a_n x^n \text{ converges} \}$ (or, sometimes, $\sum_{n=0} \ldots$)

<u>3 Examples</u> (all use the ratio test)

① $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ (here $a_n = \frac{1}{n^2}$). The ratio $\left| \frac{a_{n+1} x^{n+1}}{a_n x^n} \right| = \left(\frac{n}{n+1} \right)^2 \cdot |x| \to |x|$, so if $|x| < 1$, the series converges and if $|x| > 1$, it diverges. The endpoints, or special values, are $x = \pm 1$; the series converges for $x = 1$ by the p-test and for $x = -1$ by the alternating series test. Thus the "set of convergence" is $[-1, 1]$.

② $\sum_{n=1}^{\infty} \frac{x^n}{n!}$ (here $a_n = \frac{1}{n!}$) converges $\forall x \in \mathbb{R}$, by the ratio test. "Set of convergence" = $\mathbb{R}$.

③ $\sum_{n=1}^{\infty} n! \cdot x^n$ (here $a_n = n!$) converges ONLY for $x = 0$, again by the ratio test. "Set of convergence" = $\{0\}$.

Note: As we'll see, these 3 types of answers ($\{0\}$, $\mathbb{R}$, or an interval) are the only possibilities.

Key result: <u>Lemma.</u> Consider a power series $\sum_{n=1}^{\infty} a_n x^n$, and let $x_1 \in \mathbb{R}$.

(1) If $\sum_{n=1}^{\infty} a_n x_1^n$ converges, then $\sum_{n=1}^{\infty} a_n x^n$ converges absolutely for all $x \in \mathbb{R}$ such that $|x| < |x_1|$.

(2) If $\sum_{n=1}^{\infty} a_n x_1^n$ diverges, then $\sum_{n=1}^{\infty} a_n x^n$ diverges for all $x \in \mathbb{R}$ with $|x| > x_1$.

We leave (2) as an exercise (or see lecture notes), and prove (1).

Proof of (1) of Lemma: Assume $\sum_{n=1}^{\infty} a_n x_1^n$ converges. Then $a_n x_1^n \to 0$, and so $|a_n x_1^n| \to 0$; thus (taking $\varepsilon = 1$) $\exists\, n_0 \in \mathbb{N}$ such that if $n \geq n_0$, then $|a_n x_1^n| < 1$. Now suppose $x \in \mathbb{R}$ is such that $|x| < |x_1|$ (that is, $\left|\frac{x}{x_1}\right| < 1$). We then have, for all $n \in \mathbb{N}$, $\left|\frac{x^n}{x_1^n}\right| = \left|\frac{x}{x_1}\right|^n < 1$.

It follows that if $n \geq n_0$, $|a_n x^n| = \left|a_n x_1^n \cdot \frac{x^n}{x_1^n}\right| = |a_n x_1^n| \cdot \left|\frac{x^n}{x_1^n}\right| \underset{\substack{\text{since} \\ |a_n x_1^n| < 1}}{<} \left|\frac{x^n}{x_1^n}\right|$. Since $\left|\frac{x}{x_1}\right|^n < 1$, $\sum_{n=n_0}^{\infty} \left|\frac{x}{x_1}\right|^n$ is a convergent geometric series;

but then $\sum_{n=n_0}^{\infty} |a_n x^n|$ (and hence $\sum_{n=1}^{\infty} |a_n x^n|$) converges by the comparison test. ▨

Now look again at $A = \{x \in \mathbb{R} \mid \sum_{n=1}^{\infty} a_n x^n \text{ converges}\}$. $0 \in A$ for any power series. Suppose now that $A \neq \{0\}$, $A \neq \mathbb{R}$, and thus that A is bounded: let $R = \sup A$. By (1) of the lemma, if $|x| < R$ ($\Rightarrow \exists\, x_1 \in A$ with $|x| < x_1$), $\sum a_n x^n$ converges absolutely, so $(-R, R) \subset A$. On the other hand, by (2) of the lemma, if $|x| > R$ ($\Rightarrow \exists\, x_1 \notin A$ with $|x| > x_1 > R$), $\sum a_n x^n$ diverges, so $A \subset [-R, R]$. We thus have $(-R, R) \subset A \subset [-R, R]$, and so ...

[for ANY power series $\sum_{n=1}^{\infty} a_n x^n$; $A = \{0\}$, $A = \mathbb{R}$, or $A =$ an interval $(-R, R)$ together with one, both, or neither endpoint.]

Def. A is the interval of convergence, and R the radius of convergence (possibly $R = \infty$) of the power series $\sum a_n x^n$.

Examples

1. $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$. Using the ratio test, $\left|\frac{a_{n+1} x^{n+1}}{a_n x^n}\right| = \sqrt{\frac{n}{n+1}} \cdot |x| \longrightarrow |x|$, so $R = 1$: the power series converges for $|x| < 1$ and diverges for $|x| > 1$. The endpoints must be checked separately: if $x = 1$, we get $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ which diverges by the p-test, and if $x = -1$, we get $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ which converges by the alternating series test. Thus, the interval of convergence is $A = [-1, 1)$.

2. Suppose $\sum_{n=1}^{\infty} b_n x^n$ has radius of convergence $R = 3$. What do we know about (a) $\sum_{n=1}^{\infty} b_n 3^n$; (b) $\sum_{n=1}^{\infty} b_n (-2)^n$; (c) $\sum_{n=1}^{\infty} b_n$; (d) $\sum_{n=1}^{\infty} b_n \pi^n$?

   (ANSWERS: (a) no information; (b) converges since $|-2| = 2 < R = 3$; (c) converges, since $\sum b_n$ can be written $\sum b_n \cdot (1)^n$ and $|1| < R = 3$; (d) diverges since $|\pi| > R = 3$.)

3. Suppose $\sum_{n=1}^{\infty} c_n x^n$ converges for all $x < 0$. What is R?

   (ANSWER: $R = \infty$.)