HOMEWORK #2
SOLUTIONS

PART A

1a) The roots of $x^2+2x=0$ are $x=0$ & $x=-2$. $\forall x \in (-2,0)$ $x^2+2x<0$. For every $x<-2$, $x^2+2x>0$. Thus
$\{x \in \mathbb{R} \mid x<0 \ \& \ x^2+2x>0\} = (-\infty, -2)$.
We see that the set is not bounded below, but is bounded above with supremum -2.

b) Notice that $\{\frac{1}{2}, -\frac{1}{2}, \frac{2}{3}, -\frac{2}{3}, \frac{3}{4}, -\frac{3}{4}, \dots\}$

$$= \left\{\frac{n}{n+1} \,\middle|\, n \in \mathbb{N}\right\} \cup \left\{-\frac{n}{n+1} \,\middle|\, n \in \mathbb{N}\right\}.$$

This set is bounded above & below.
The supremum is 1 & the infimum is -1.
To prove it let $x>0$. Then by theorem 2.14 (I think) $\exists\, n_0 \in \mathbb{N}$ s.t. $0<\frac{1}{n_0}<x$. It follows that $1-x < 1-\frac{1}{n_0} = \frac{n_0-1}{n_0}$, & $-1+x > -1+\frac{1}{n_0} = -\frac{n_0-1}{n_0}$. Thus given any number y less than 1 there is an element of the set greater than y & likewise for the infimum.

c) We have $\{1-.9, 1-.99, 1-.999, \dots\}$

$$= \{.1, .01, .001, \dots\} = \left\{\frac{1}{10^n} \,\middle|\, n \in \mathbb{N}\right\}.$$

This set is bounded with infimum 0 & supremum .1.

2. a) Since $A \subset \{\sin(x) \mid x \in \mathbb{R}\} = [-1,1]$, it follows that its bounded.

b) Notice $\frac{\pi}{4} < \frac{\pi}{2} < \frac{3\pi}{2} < \frac{7\pi}{4}$.

Thus $\sin(\pi/2) = 1 \in A$, & $\sin(\frac{3}{2}\pi) = -1 \in A$.

Since $A \subset [-1,1]$ we have $\sup(A) = 1$ & $\inf(A) = -1$.

c) By part b) both $\sup(A)$ & $\inf(A)$ are elements of A.

3 a) Pf: Let R be an upper bound for A,

Then $x \leq R \;\forall\, x \in A$, & thus $-R \leq -x \;\forall x \in A$.
Since $-x \in -A \;\forall\, x \in A$ we find $-R \leq y \;\forall\, y \in -A$.
That is, $-A$ is bounded below,

b) Pf: By definition, w is an upper bound of A.

Thus by 3a) $-w$ is a lower bound for $-A$.
Let $-r$ be any lower bound for $-A$. Then
r is an upper bound for A since $-r \leq -x \;\forall\, -x \in -A$
& so $r \geq x \;\forall\, x \in A$. Since $w = \sup A$, $r \geq w$,
& thus $-r \leq -w$. That is given a lower bound $-r$
for $-A$, $-r \leq -w$. $\therefore$ $-w = \inf(-A)$,

4. We're given that $B \subset A$. To avoid messiness let $\inf(A) = -\infty$ when A isn't bounded below.

Claim: $\inf(A) \leq \inf(B)$.

Pf of claim: By definition, $B \subset A \iff \forall x \in B,\ x \in A$.

let $r = \inf(A)$ [r may be $-\infty$]. Since $r \leq x\ \forall x \in A$ it follows that $r \leq x\ \forall x \in B$. That is, r is a lower bound for B. By definition, if $r' = \inf B$, $r' \geq s$ for any lower bound s of B, including r. i.e. $r \leq r'$. That is $\inf(A) \leq \inf(B)$.

PART B:

5. First observe: $4^k > k^4 \iff (2^2)^k > (k^2)^2 \iff (2^k)^2 > (k^2)^2$

$\iff 2^k > k^2$. That is, $\forall k \in \mathbb{N}\ [4^k > k^4] \iff [2^k > k^2]$.

Let's investigate 2^k & k^2 for some k's:

k	2^k	k^2
1	2	1
2	4	4
3	8	9
4	16	16
5	32	25.

Claim: $2^k > k^2\ \forall k \geq 5$.

Pf of claim: (By induction)

Base case: Let $k=5$. Then $2^k = 32 > 25 = 5^2$.

5 cont.

Inductive step: Suppose for some $k \geq 5$, $2^k > k^2$.

Then we have

$$2^{k+1} = 2 \cdot 2^k > 2k^2.$$

So if we can prove that $2k^2 \geq (k+1)^2 \quad \forall\, k \geq 5$ then it will follow that $2^{k+1} > 2k^2 \geq (k+1)^2$ & the proof will be complete.

Well

$$2k^2 \geq (k+1)^2$$

$$\Longleftrightarrow \quad 2k^2 \geq k^2 + 2k + 1$$

$$\Longleftrightarrow \quad k^2 - 2k - 1 \geq 0$$

The roots of $x^2 - 2x - 1 = 0$ are $x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(-1)}}{2} = 1 \pm \sqrt{2}$.

If $x > 1 + \sqrt{2}$ we have $x^2 - 2x - 1 > 0$.

Since $k \geq 5$ we have $k^2 - 2k - 1 > 0$ & so $2k^2 \geq (k+1)^2$, & $2^{k+1} > (k+1)^2$.

We have proven that $\forall k \geq 5$, $2^k > k^2$ & by our initial observations we conclude $\forall k \geq 5$, $4^k > k^4$, as desired.

6a) The $(2k-1)^{st}$ element of A can be written

$$2+\sum_{j=1}^{k}\frac{1}{10^j} = 2+\sum_{j=1}^{k}\left(\frac{1}{10}\right)^j$$

& the $(2k)^{th}$ element of A has the form

$$-2-\sum_{j=1}^{k}\frac{3}{10^j} = -2-3\sum_{j=1}^{k}\left(\frac{1}{10}\right)^j$$

b) It is clear that every odd element is greater than 2 & every even element is less than -2. Hence an upper bound of the odd elements will be an upper bound for A & likewise a lower bound of the even elements gives us a lower bound for A.

From class we know $\sum_{j=1}^{k}\left(\frac{1}{10}\right)^j = \dfrac{\frac{1}{10}-\left(\frac{1}{10}\right)^{k+1}}{1-\frac{1}{10}} = \dfrac{\frac{1}{10}-\frac{1}{10^{k+1}}}{\frac{9}{10}}$

$< \frac{1}{9} \quad \forall k.$

$\therefore \quad B = [2+\frac{1}{9}, \infty)$ & $C = (-\infty, -2-3\cdot\frac{1}{9}]$.

c&d) The $\inf A = \sup C = -2-\frac{1}{3}$ & $\sup A = \inf B = 2+\frac{1}{9}$.
It is clear from how we got these values that both are indeed bounds, so to prove they are $\inf(A)$ & $\sup(A)$ we need only show that given any lower bound r of A $r \leq -2-\frac{1}{3}$ & given R an upper bound of A $R \geq 2+\frac{1}{9}$.

6c&d) cont.

First notice that given any $n \in \mathbb{N}$, $10^n \geq n$.

This should be obvious. By theorem 2.14 (I think)

Given any $x > 0$ $\exists$ $n_0 \in \mathbb{N}$ s.t. $\frac{1}{n_0} < x$, & since

$n_0 \leq 10^{n_0}$ we find $\frac{1}{10^{n_0}} < x$ also.

Now say $r > -2-\frac{1}{3}$ & r is a lower bound

for A. Let $x = r-(-2-\frac{1}{3}) > 0$, & let $n_0 \in \mathbb{N}$

be s.t. $\frac{1}{10^{n_0}} < 3x$. By the above we have that

$$\left(\begin{array}{c}\text{the } 2n_0^{th} \text{ element} \\ \text{of } A\end{array}\right) = -2-3\left(\sum_{j=1}^{n_0}\left(\frac{1}{10}\right)^j\right) = -2-3\left(\frac{\frac{1}{10}-\frac{1}{10^{n_0+1}}}{\frac{9}{10}}\right)$$

$$= -2 - \frac{1}{3}\left(1-\frac{1}{10^{n_0}}\right)$$

$$< -2 - \frac{1}{3}(1-3\cdot x)$$

$$= -2 - \frac{1}{3} + x = -2-\frac{1}{3} + r - (-2-\frac{1}{3}) = r.$$

Thus r is not a lower bound for A & so $-2\frac{1}{3} =$

$-2-\frac{1}{3} = \inf(A)$.

Similarly, R is an upper bound for A with $R < 2+\frac{1}{9} = 2\frac{1}{9}$

Let $x = 2+\frac{1}{9} - R$ & let $n_0 \in \mathbb{N}$ be s.t. $\frac{1}{10^{n_0}} < 9x$.

We find that

$$(\text{the } (2n_0+1)^{st} \text{ element of } A) = 2 + \sum_{j=1}^{n_0}\left(\frac{1}{10}\right)^j = 2 + \frac{1}{9}\left(1-\frac{1}{10^{n_0}}\right)$$

$$> 2 + \frac{1}{9} - x = 2+\frac{1}{9} - (2+\frac{1}{9} - R) = R,$$

Thus R is not an upper bound for A & so $2\frac{1}{9} = \sup A$.

6e) We have $B \cup C = (-\infty, -2-\frac{1}{3}] \cup [2+\frac{1}{9}, \infty)$.

Clearly $0 \notin B \cup C$.

7. Given $a > 0$, $\inf\{\frac{a}{n} \mid n \in \mathbb{N}\} = 0$.

Pf: $\forall n \in \mathbb{N}$, $\frac{a}{n} > 0$ so 0 is a lower bound for $\{\frac{a}{n} \mid n \in \mathbb{N}\}$. Suppose $r > 0$ is ~~also~~ a lower bound for $\{\frac{a}{n} \mid n \in \mathbb{N}\}$. Since $a > 0$, $\frac{r}{a} > 0$ & hence by theorem 2.4 (I'm sure that's it). $\exists\, n_0 \in \mathbb{N}$ s.t. $\frac{1}{n_0} < \frac{r}{a}$. But then $\frac{a}{n_0}$ is an element of $\{\frac{a}{n} \mid n \in \mathbb{N}\}$ & $\frac{a}{n_0} < r$, contradicting the assumption that r was a lower bound. Thus 0 is the greatest lower bound, i.e. $0 = \inf\{\frac{a}{n} \mid n \in \mathbb{N}\}$.