Solutions to HW4

PART A

1. a) $a_n = \dfrac{\ln(n)}{n^{1/3}}$, $n \geq 1$.

Ans: By L'Hopital we have $\lim\limits_{n\to\infty} \dfrac{\ln(n)}{n^{1/3}} = \lim\limits_{n\to\infty} \dfrac{1/n}{\frac{1}{3} n^{-2/3}}$

$$= \lim_{n\to\infty} \frac{3n^{2/3}}{n} = \lim_{n\to\infty} \frac{3}{n^{1/3}} = 0.$$

b) $b_n = n \tan\left(\frac{1}{n}\right)$.

Ans: We have $n\tan\left(\frac{1}{n}\right) = \dfrac{\tan\left(\frac{1}{n}\right)}{\frac{1}{n}}$.

Now by prop. 4.1, if $\lim\limits_{x\to 0^+} \dfrac{\tan(x)}{x} = L$

then $\lim\limits_{n\to\infty} \dfrac{\tan\left(\frac{1}{n}\right)}{1/n} = L$.

By L'Hopital we have

$$\lim_{x\to 0^+} \frac{\tan(x)}{x} = \lim_{x\to 0^+} \frac{\frac{1}{\cos^2(x)}}{1} = 1.$$

Thus $b_n \to 1$.

2. a) $c_n = \dfrac{\sqrt{n}+1}{\sqrt[3]{n}+2}$.

Ans: We have $c_n = \dfrac{n^{1/2}+1}{n^{1/3}+2} = \dfrac{\frac{1}{n^{1/3}}(n^{1/2}+1)}{\frac{1}{n^{1/3}}(n^{1/3}+2)}$

$$= \frac{n^{1/6}+\frac{1}{n^{1/3}}}{1+\frac{2}{n^{1/3}}} > n^{1/6}+\frac{1}{n^{1/3}} > n^{1/6}$$

for each $n \in \mathbb{N}$. Since $\langle n^{1/6} \rangle$ is divergent to ∞ it follows that c_n must also diverge to ∞.

b) $d_n = \dfrac{\sin(n)^n}{n}$.

Ans: Since $|\sin(n)| \leq 1$ for each $n \in \mathbb{N}$, we have $|\sin(n)^n| \leq 1$. for each $n \in \mathbb{N}$.

Thus $\forall n$, $-\frac{1}{n} \leq \frac{\sin(n)^n}{n} \leq \frac{1}{n}$.

By the pinching theorem we conclude $d_n \to 0$.

3. a) Pf: We'll show $s_n \leq 6$ $\forall n \in \mathbb{N}$, by induction.

Base Case: $s_1 = \sqrt{6} < 6$.

Inductive step: Suppose for some $n \in \mathbb{N}$ $s_n \leq 6$.

Then $s_{n+1} = \sqrt{6 + s_n} \leq \sqrt{6+6}$. (since $f(x) = \sqrt{x+6}$ is an increasing function)

& $\sqrt{6+6} = \sqrt{12} < 4 < 6$. By induction

$s_n \leq 6$ $\forall n \in \mathbb{N}$.

b) Ans: We know that $\langle s_n \rangle$ is a convergent sequence because it is increasing & bounded above. (Monotone convergence theorem). Let $L = \lim_{n\to\infty} s_n$.

Then we have $L = \lim_{n\to\infty} s_{n+1} = \lim_{n\to\infty} \sqrt{6+s_n}$, & since $f(x) = \sqrt{6+x}$ is a continuous function on the interval $[-6,\infty)$ & $s_n \in [0,\infty)$ for each n theorem 3.1 gives us that $L = \lim_{n\to\infty} \sqrt{6+s_n} = \sqrt{6 + \lim_{n\to\infty}(s_n)} = \sqrt{6+L}$.

Thus, $L = \sqrt{6+L} \iff L^2 = 6 + L \iff L^2 - L - 6 = 0$

$\iff (L-3)(L+2) = 0$.

So $L = 3$ or $L = -2$. But $\langle s_n \rangle$ is increasing & starts positive so $L = 3$.

4.a) Following the hint let

$f(x) = \frac{1}{x} - a$. Then $f(\frac{1}{a}) = \frac{1}{\frac{1}{a}} - a = 0$. We may use Newton's method to get a sequence which converges to the root of f, (i.e. $\frac{1}{a}$).

Now $f'(x) = -\frac{1}{x^2}$. Let $x_1 = c$ for some $c > 0$ & define

$$\underline{x_{n+1}} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n + \frac{\frac{1}{x_n} - a}{\frac{1}{x_n^2}} = x_n + x_n^2\left(\frac{1}{x_n} - a\right)$$

$$= x_n + x_n - a x_n^2 = 2x_n - a x_n^2$$

b) Lets take $x_1 = \frac{1}{16}$. Then $x_2 = \frac{1}{8} - 17\left(\frac{1}{16^2}\right) = \frac{1}{8} - \frac{17}{256} = \frac{15}{256}$

& $x_3 = 2\left(\frac{15}{256}\right) - 17\left(\frac{225}{65536}\right) \approx .0588226$.

So $\frac{1}{17} - x_3 \approx 8.98 \times 10^{-7}$.

Let $\varepsilon > 0$.

5. Pf: By definition $\lim_{x \to 0^+} f(x) = L$ iff given ε, $\exists\ \delta > 0$ s.t.

$0 \le y < \delta \Rightarrow |f(y) - L| < \varepsilon$.

We know that given $\delta > 0$ $\exists\ n_0 \in \mathbb{N}$ s.t. $m \geq n_0 \Rightarrow \frac{1}{m} < \delta$.

Thus given $\varepsilon > 0$ $\exists\ n_0 \in \mathbb{N}$ s.t. $m \geq n_0 \Rightarrow$

$$\left(\frac{1}{m} < \delta \Rightarrow\right)$$

$$\left|f\left(\frac{1}{m}\right) - L\right| < \varepsilon.$$

$\therefore \lim_{n \to \infty} f\left(\frac{1}{n}\right) = L$.

6. a) Pf: We'll show in part (b) that the sequence is increasing. It follows that $1 \le s_n$ $\forall\ n \in \mathbb{N}$ since $s_1 = \sqrt{5} > 1$.

We have that $s_1 = \sqrt{5} < 5$.

Now assume $s_n < 5$ for some $n \in \mathbb{N}$. Then $s_{n+1} = \sqrt{5 s_n} < \sqrt{5 \cdot 5} = 5$. By induction, we have shown that $s_n \le 5$ $\forall\ n \in \mathbb{N}$.

6 b) Pf: We'll show that $\langle s_n \rangle$ is increasing by induction.

Base Case: We have

$$s_2 = \sqrt{5 s_1} = \sqrt{5\sqrt{5}} = \sqrt{5}\sqrt{\sqrt{5}} = s_1\sqrt{\sqrt{5}} > s_1,$$

since $\sqrt{\sqrt{5}} > 1$.

Now assume that $s_n > s_{n-1}$ for some $n \geq 2$.

We have $s_{n+1} = \sqrt{5 s_n} > \sqrt{5 s_{n-1}} = s_n$.

Thus s_n is increasing.

c) We know that $\langle s_n \rangle$ converges because it is bounded & increasing (Monotone convergence theorem). Let $L = \lim_{n\to\infty} s_n$. Then $L = \lim_{n\to\infty} s_{n+1}$ (because $\langle s_n \rangle$ & $\langle s_{n+1} \rangle$ have the same limit when $\langle s_n \rangle$ converges).

& so $L = \lim_{n\to\infty} \sqrt{5 s_n}$. Since the function $f(x) = \sqrt{5x}$ is continuous over the interval $(0,\infty)$ theorem 3.1 gives us $\lim_{n\to\infty}\sqrt{5 s_n} = \sqrt{5 \lim_{n\to\infty} s_n} = \sqrt{5L}$.

Thus $L = \sqrt{5L} \Rightarrow L^2 = 5L \Rightarrow L(L-5) = 0 \Rightarrow L = 0$ or $L = 5$

Since $s_n \in [1,5]$ $\forall$ $n \in \mathbb{N}$ we conclude that $\lim_{n\to\infty} s_n = 5$.

7. a) <u>Pf</u>: We have $a_n = \dfrac{n^2-4n-5}{n^2-2n-3}$

$$= \frac{(n-5)(n+1)}{(n-3)(n+1)}$$

$$= \frac{n-5}{n-3}$$

$$= \frac{\frac{1}{n}(n-5)}{\frac{1}{n}(n-3)}$$

$$= \frac{1-\frac{5}{n}}{1-3/n}.$$

It is clear that $1-5/n \to 1$ & $1-\frac{3}{n} \to 1$.
(Note: a_3 isn't defined, but for every $n \neq 3$, $1-\frac{3}{n} \neq 0$).
By theorem 2.1 (d) we have $a_n \to 1$.

b) Well $|a_n - 1| < \frac{1}{100} \iff \left|\frac{n-5}{n-3} - 1\right| < \frac{1}{100}$

$$\iff \left|\frac{n-5-n+3}{n-3}\right| < \frac{1}{100}$$

$$\iff \left|\frac{-2}{n-3}\right| < \frac{1}{100}$$

Clearly n needs to be s.t. $n>3$ & so we want

$$\frac{2}{n-3} < \frac{1}{100} \iff 200 < n-3 \iff 203 < n.$$

Thus $n \geq 204 \Rightarrow |a_n - L| < \frac{1}{100}$.