Homework #3 for MATH 8302: Manifolds and Topology II

March 20, 2018

Due Date: Monday 26 March in class.

1. Let $f : X \to Y$ be a smooth submersion between two smooth, compact manifolds of the same dimension. Show that $f : X \to Y$ is a covering space.

2. Fix positive integers n and k, with $k \leq n$.

   (a) Show that the set $S \subset (\mathbb{R}^n)^{\times k}$ consisting of all linearly independent k-tuples $(v_1, \ldots, v_k)$ of vectors $v_j \in \mathbb{R}^n$ forms an open subset.[1]

   (b) Show that the map $\sigma : \mathbb{R}^k \times S \to \mathbb{R}^n$ given by

   $$[(t_1, \ldots, t_k), (v_1, \ldots, v_k)] \mapsto t_1 v_1 + \cdots + t_k v_k$$

   is a submersion.

   (c) There is an action of the group $\mathrm{GL}_k(\mathbb{R})$ on S, where for a matrix $A = (a_{ij}) \in \mathrm{GL}_k(\mathbb{R})$

   $$A \cdot (v_1, \ldots, v_k) = (\sum_j a_{1j} v_j, \ldots, \sum_j a_{kj} v_j)$$

   Construct a bijection from the set of orbits $\mathrm{GL}_k(\mathbb{R}) \backslash S$ to the set G of subspaces of $\mathbb{R}^n$ of dimension k.

   (d) Let X be a submanifold of $\mathbb{R}^n$. Prove that there is a dense subset of $T \subseteq S$ with the property that if $(v_1, \ldots, v_k) \in T$, then X intersects the span $V = \langle v_1, \ldots, v_k \rangle$ transversally. Colloquially: almost every k-dimensional subspace $V \leq \mathbb{R}^n$ intersects X transversally.

   (e) **(Bonus problem, not required)** The space G is called the *Grassmannian* of k-planes in $\mathbb{R}^n$; part (c) allows us to topologize G via the quotient topology on $\mathrm{GL}_k(\mathbb{R}) \backslash S$. Show that G is a manifold of dimension $(n-k)k$.

[1] The space S is a slight variant on the *Stiefel manifold*, where the v_j are required to additionally be orthonormal.

3. Let $f : V \to W$ be a linear map. Picking a basis $v_1, \ldots, v_n$ and $w_1, \ldots, w_m$ of V and W, respectively, the matrix for f is given by $A = (a_{ij})$, where

$$f(v_i) = \sum_j a_{ij} w_j$$

(a) A basis for $\Lambda^p V$ is given by $v_{i_1} \wedge \cdots \wedge v_{i_p}$, where $1 \leq i_1 < i_2 < \cdots < i_p \leq n$. Compute the matrix of $\Lambda^2 f$ with respect to this basis (when $p = 2$); if you're feeling excited, extend this to arbitrary p.

(b) Prove that the map $\mathrm{Hom}(V, W) \to \mathrm{Hom}(\Lambda^2 V, \Lambda^2 W)$ which carries f to $\Lambda^2 f$ is smooth. Here, we use the fact that $\mathrm{Hom}(V, W) \cong \mathbb{R}^{nm}$ to define smoothness.