Homework #4 for MATH 5345H: Introduction to Topology

October 2, 2019

Due Date: Friday 4 October in class.

Focus on writing: Writing strong mathematical proofs is just as much about quality writing as it is about quality content. Over the next few weeks, in addition to writing up solutions to your problem sets as usual, I will ask you to focus intently on improving one aspect of your proof writing skills.

Structuring quality proofs takes careful thought; proofs should not be written in stream-of-conscience form. Writing out one's train of thoughts is a great way to build a proof sketch, and it needs to be honed from there into an argument that follows a logical roadmap. Writing proofs with an intended structure in mind also improves the concision of your proofs.

Before writing your solution to Problem 4, I want you two write an outline to your argument that details the purpose of each section.

For example, a problem might ask you to prove that the group of integers is isomorphic to the group of even integers. An appropriate outline might be the following:

1. Define a map $\varphi : \mathbb{Z} \to 2\mathbb{Z}$
2. Prove φ is a bijection
   (a) Prove φ is injective
   (b) Prove φ is surjective
3. Prove φ is a homomorphism

After constructing your outline, you should should write your proof to follow it.

Problems:

1. Show that the dictionary order topology on $\mathbb{R} \times \mathbb{R}$ is the same as the product topology on $\mathbb{R}_d \times \mathbb{R}$, where $\mathbb{R}_d$ denotes $\mathbb{R}$ with the discrete topology.
2. Show that the collection
$$\{(a, b) \times (c, d) \mid a, b, c, d, \in \mathbb{Q},\ a < b,\ c < d\}$$
is a countable basis for the product topology on $\mathbb{R}^2$.

3. Let X be an ordered set, and give it the order topology. Show that the closure $\overline{(a,b)}$ of the open interval (a,b) is contained in the closed interval $[a,b]$; that is, $\overline{(a,b)} \subseteq [a,b]$. Under what conditions (i.e., for what properties of the order) are they equal?

4. Regard the real line $\mathbb{R}$ as a subset of the plane $\mathbb{R} \times \mathbb{R}$: we will identify $\mathbb{R}$ with the subset

$$\mathbb{R} \times \{0\} = \{(r,0), r \in \mathbb{R}\} \subseteq \mathbb{R} \times \mathbb{R}.$$

Equip $\mathbb{R} \times \mathbb{R}$ with the Zariski topology as defined in the previous homework: closed sets $V(S)$ are the common zero locus of sets S of polynomials $f(x,y)$ in two variables.

Show that the subspace topology on $\mathbb{R} \times \{0\}$ is "the same" as the Zariski topology on $\mathbb{R}$. More concretely, let

$$f : \mathbb{R} \times \{0\} \to \mathbb{R} \text{ and } f^{-1} : \mathbb{R} \to \mathbb{R} \times \{0\}$$

be the inverse functions $f(x,0) = x$ and $f^{-1}(x) = (x,0)$. Show that $U \subseteq \mathbb{R}$ is open in the Zariski topology on $\mathbb{R}$ if and only if $f^{-1}(U)$ is open in $\mathbb{R} \times \{0\}$, regarded as a subspace of $\mathbb{R} \times \mathbb{R}$ with the Zariski topology.