Midterm for MATH 5345H: Introduction to Topology

October 15, 2018

Due Date: Monday 22 October in class. You may use your book, notes, and old homeworks for this exam. When using results from any of these sources, please cite the result being used. Please explain all of your arguments carefully. Please do not communicate with other students about the exam. You are free to contact me with questions about the exam at any time.

1. Let X and Y be topological spaces, and let $f : X \to Y$ be a function. Define

$$Z := \{f(x) \mid x \in X\} \subseteq Y$$

to be the image of the function f, and equip Z with the topology as a subspace of Y. Define $\overline{f} : X \to Z$ by $\overline{f}(x) = f(x)$. Show that f is continuous if and only if $\overline{f}$ is continuous.

2. Let $\mathbb{Q}$ denote the rational numbers.

   (a) Let T_{order} denote the order topology on $\mathbb{Q}$ with respect to the usual $<$, and let $T_{subspace}$ denote the subspace topology on $\mathbb{Q}$, as a subspace of $\mathbb{R}$ with the standard topology. Show that $T_{order} = T_{subspace}$.

   (b) Are the two topologies in the previous problem the same as the discrete topology? Why or why not?

3. Give $\mathbb{R}$ the standard topology, and let $A = \mathbb{Q}$, regarded as a subset of $\mathbb{R}$. Show that $\overline{A} = \mathbb{R}$. What are the set of limit points of A?

4. Let $Y = \{a, b\}$ be a set with two elements, let X be any set, and let $P(X)$ be the power set of X (i.e., the set of subsets of X).

   (a) Write $Y^X := \{f : X \to Y\}$ for the set of functions from X to Y. Show that the function

$$\alpha : Y^X \to P(X) \text{ given by } \alpha(f) = f^{-1}(a)$$

   is a bijection.

   (b) Let $T_Y = \{\emptyset, \{a\}, Y\} \subseteq P(Y)$. I claim that T_Y is a topology on Y – this is easy to verify, and you do not need to do so.

   Now assume, further, that X has a topology T_X. Prove that α restricts to a bijection

$$\alpha : \{f : X \to Y \mid f \text{ is continuous}\} \to T_X = \{U \subseteq X \mid U \text{ is open}\}.$$

5. A subset $A \subseteq X$ is said to be *dense* if $\overline{A} = X$. If $A \subseteq X$ and $B \subseteq Y$ are both dense, show that $A \times B$ is dense in $X \times Y$ (in the product topology).