# Homework #11 for MATH 5345H: Introduction to Topology

December 4, 2017

**Due Date:** Monday 11 December in class.

1. Let $X$ be a topological space, and $A \subseteq X$ a subspace. $A$ is said to be a *retract* of $X$ if there exists a continuous function $p : X \to A$ with the property that $p(a) = a$ for every $a \in A$.

   (a) If $A$ is a retract of $X$, show that the inclusion map $i : A \to X$ (given by $i(a) = a$) induces an injective homomorphism $i_* : \pi_1(A, a_0) \to \pi_1(X, a_0)$ for any $a_0 \in A$. Further, show that $p_* : \pi_1(X, a_0) \to \pi_1(A, a_0)$ is surjective.

   (b) The map $p$ is said to be a *deformation retraction* if, further, there is a homotopy $i \circ p \simeq \mathrm{id}_X$ relative to $A$ (that is, if $H$ is the homotopy, then $H(a, t) = a$ for every $a \in A$). Show that in this case, $p_*$ and $i_*$ are inverse isomorphisms.

2. A region $P \subseteq \mathbb{R}^2$ in the plane is said to be *star-shaped* with respect to a point $p \in P$ if for every $q \in p$, the straight line $\overline{pq}$ from $p$ to $q$ is contained in $P$.

   (a) Show that if $P$ is star-shaped, then it is *contractible* (homotopy equivalent to a point).

   (b) If $P$ is a polygon which is star-shaped with respect to a point $p$ in the interior of $P$, define a function

   $$f : P \setminus \{p\} \to S^1 \text{ via } f(q) = \frac{q - p}{|q - p|}$$

   and show that $f$ is a homotopy equivalence.

   (c) Let $P$ be the square $[-1, 1] \times [-1, 1] \subseteq \mathbb{R}^2$, and let $T^2$ be the *torus*: $T^2$ is the quotient space $P/\sim$ under the relation $\sim$ which identifies the left side with the right, and the top with the bottom:

   $$(-1, y) \sim (1, y) \text{ and } (x, -1) \sim (x, 1).$$

Prove that if $p \in T^2$ is the image of $(0,0) \in P$, then there is a homotopy equivalence

$$T^2 \setminus p \simeq S^1 \vee S^1$$

from the torus punctured at $p$ to the wedge of two circles (here, the *wedge* of spaces $X$ and $Y$ with respect to two points $x \in X$ and $y \in Y$ is $X \vee Y := X \sqcup Y / \sim'$, where $\sim'$ is the equivalence relation which *only* identifies $x \sim' y$.