Special case (Ahlfors 6.2.3)

n-gon is rectangle. Can pick two of x_i arbitrarily in Schwarz-Christoffel.

Set $x_1 = 0$, $x_2 = 1$, $x_3 := \rho$ some real # > 1, $x_4 = \{\infty\}$.

Map from $\mathcal{H}$ to rectangle takes form: (exterior angles $\pi \alpha_i = \pi/2$)
i.e. $\alpha_i = 1/2$

$$f(z) = \int_{z_0}^{z} (\xi - 0)^{-1/2} (\xi - 1)^{-1/2} (\xi - \rho)^{-1/2} \, d\xi$$

$$= \int_{z_0}^{z} \frac{d\xi}{\sqrt{\xi(\xi-1)(\xi-\rho)}}$$ "elliptic integral"

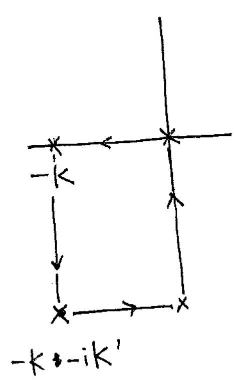
Ahlfors takes $z_0 = 0$ as improper integral (limit as $z_0 \to 0$)

As z traverses x-axis, see that for $z \in (0,1)$ $\sqrt{\xi}$ real, positive (choice of branch)
$\sqrt{\xi - 1}$, $\sqrt{\xi - \rho}$ imag.
so integrand is negative.

Let $\displaystyle\int_0^1 \frac{d\xi}{\sqrt{\xi(\xi-1)(\xi-\rho)}} =: -K$, K: pos. real.

then from $(1, \rho)$, $\sqrt{\xi}$, $\sqrt{\xi - 1}$ real, $\sqrt{\xi-\rho}$ imag., so contribution is pure imag., in denom, ie neg. imag. in numerator

write $\displaystyle\int_1^{\rho} (-''-) = -K - iK'$



Remains: $\displaystyle\int_{\rho}^{\infty} (-''-)$ and $\displaystyle\int_{-\infty}^{0} (-''-)$

Use Cauchy's thm to show $\displaystyle\int_{-\infty}^{\infty} (-''-) = 0$

using semicircular contour, avoiding poles.

Hence, must get back "home"

Cool fact: use reflection principle to create doubly periodic function.

Solve Dirichlet problem on simply-conn. open set

Do this by transferring the problem to the unit disk, solving it there.

To do this, formulate a maximum and minimum principle for harmonic functions (harmonic functions are real-valued, so if we have maximum principle for $u(x,y)$, then apply that to $-u(x,y)$ to get minimum principle.

Contrast this with f: analytic, then discuss max of $|f|$. No way to obtain similar minimum thm. — can consider $|1/f|$, but this is problematic if $f=0$.)

To prove maximum principle for harmonic functions, go back to that for analytic ones: Two proofs: topological – corollary of open mapping thm which followed from winding # result

analytic – Cauchy integral formula $\gamma = C(z_0; r)$

parametrize ξ on curve $C(z_0;r)$ as usual by $\xi = z_0 + re^{i\theta}$

$d\xi = ire^{i\theta}\,d\theta$

then C.I.F. reads:

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})\,d\theta$$

so f is the arithmetic mean of values on $C(z_0;r)$ provided we choose r small enough to remain in f's domain of analyticity.

$$\Rightarrow \quad |f(z_0)| = \frac{1}{2\pi}\left|\int_0^{2\pi} f(z_0+re^{i\theta})\,d\theta\right| \leq \frac{1}{2\pi}\int_0^{2\pi} |f(z_0+re^{i\theta})|\,d\theta \qquad (*)$$

if $|f(z_0)|$ is max for $|f(z)|$, then $|f(z_0+re^{i\theta})| \leq |f(z_0)|$.

If inequality is strict for some θ, then strict on open interval by continuity and this contradicts $(*)$. So must have

equality $|f(z_0)| = |f(z_0+re^{i\theta})|$ on all sufficiently small circles
i.e. r small enough, all θ

$\Rightarrow$ $f(z)$ constant in nbhd of z_0

$\Rightarrow$ $f(z)$ constant on domain of def'n //

Want similar "mean-value property" for harmonic functions, then proof is just the same as above after that. Indeed, this is true writing $z_0 = x_0 + iy_0$:

$$u(z_0) := u(x_0, y_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0+re^{i\theta})\, d\theta$$

pf: Pick simply-connected nbhd. of $C(z_0;r)$, so that u is $\mathrm{Re}(f)$ for some analytic f on nbhd. Now take real parts of both sides of mean-value thm for analytic functions. //

Thm (uniqueness of Dirichlet problem) For any region Ω:

pf: Given two sol'ns $u, \tilde{u}$, then $\phi := u - \tilde{u}$ is harmonic and $= 0$ on $\partial\Omega$. Now max/min principles for harmonic functions $\Rightarrow$ if ϕ constant on $\partial\Omega$, ϕ constant on Ω (and in this case $= 0$) //

To solve the Dirichlet problem, basically done since mean value thm gives

$$u(0) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\, d\theta.$$

Now use linear transformations to map any point of disk to origin