Sketch <u>of pf</u> : simple pole for f at p means, in local coords, say z centered at p,

$$f(z) = a_{-1} z^{-1} + a_0 + a_1 z + \cdots$$

If $\tilde{z}$ is another local coord centered at p, then $\tilde{z} = c_1 z + c_2 z^2 + \cdots$

so $\tilde{a}_{-1} = c_1^{-1} a_{-1}$. i.e. $a_{-1} \cdot \frac{\partial}{\partial z}$ defines tangent vector at p independent of choice of coord.

Thus residue at $p \longleftrightarrow$ elt. of TX_p.

So have map $\mathrm{Res}: H^0(D) \longrightarrow \bigoplus_i TX_{p_i}$

$$f \longmapsto \left(\mathrm{res}_{p_1}(f), \ldots, \mathrm{res}_{p_d}(f) \right)$$

kernel is holomorphic functions on X, but since X compact these are constant.

At each TX_{p_i} we have associated $(0,1)$ form $\alpha_i = \bar{\partial}\left(\underset{\sim}{\beta} \frac{1}{z} \right)$

comp. supp. function in nbhd. of p_i

$\longleftrightarrow$ linear map $A_i : TX_{p_i} \longrightarrow H^{0,1}$

let $\underline{A} = (A_1, \ldots, A_d)$ so $\underline{A} : \bigoplus_i TX_{p_i} \to H^{0,1}$

See subsequent pages for clarification of sketch below the line

<u>claim</u> : The following is an exact sequence :

$$0 \to \mathbb{C} \to H^0(D) \to \bigoplus_i TX_{p_i} \to H^{0,1}$$

<u>corollary</u> : By rank-nullity thm. $\dim \ker \underline{A} = d - \dim H^{0,1} + \dim \ker \underline{A}^T$

<u>claim 2</u> : $\dim \ker \underline{A} = h^0(D) - 1$, $\dim \ker \underline{A}^T = h^0(K-D)$.

Here $\underline{A}^T$ acts on dual spaces $(H^{0,1})^* \longrightarrow \bigoplus_i TX_{p_i}^*$

Recall that two lectures ago, having simple pole at $p \in X \iff$

in local coord z centered at p, $\beta \cdot \frac{1}{z}$ merom. function. β : smooth, comp. supp. locally const. at 0

And $\bar{\partial}\left(\beta \frac{1}{z}\right) = \underbrace{\bar{\partial}\beta \cdot \frac{1}{z}}_{A}$ is a smooth $(0,1)$-form on X

From our coord-free point of view: A is linear map from $TX_p \to H^{0,1}$

Similarly extend to case of d simple poles with linear map

$$\underline{A} : \bigoplus_i TX_{p_i} \longrightarrow H^{0,1}$$

<u>claim</u>: The following is an exact sequence:

$$0 \longrightarrow \mathbb{C} \xrightarrow{\text{const.}} H^0(D) \xrightarrow{\text{Res}} \bigoplus_i TX_{p_i} \xrightarrow{\underline{A}} H^{0,1}$$

just need to show $\operatorname{Im}(\text{Res}) = \ker(\underline{A})$.

this is our previous assertion. for single simple pole. Not hard to modify for d poles.

so if we want a function f on X with simple pole at p, find smooth g on X s.t. $g + \beta\frac{1}{z}$ holom. on $X \setminus \{p\}$.

i.e. solve $\bar{\partial} g = -A$

i.e. show $[A]$ in $H^{0,1}$ $= 0$.

From exact sequence, we have: (identity of cx. vector spaces)

$$\dim \ker \underline{A} = d - \dim H^{0,1} + \dim \ker \underline{A}^T \qquad (*)$$

(rank-nullity thm in linear algebra, plus fact that

$$\text{rank} = \dim \operatorname{Im} A = \dim H^{0,1} - \underbrace{\dim \ker \underline{A}^T}_{\text{"left null space"}}$$

which is orthogonal complement of column space of $\underline{A}$.)

Strictly speaking, $\underline{A}^T$ acts on dual space.

(Given linear transformation $L : V_1 \to V_2$, then $L^* : V_2^* \to V_1^*$

$\phi \mapsto \phi \circ L$

and if we give a basis of V_1, V_2 so that L is rep'd by $\underline{A}$, then $L^* = \underline{A}^T$.)

so $\underline{A}^T : (H^{0,1})^* \to \bigoplus_i (TX_{p_i})^*$.

By corollary of main thm, $(H^{0,1})^* \simeq H^{1,0}$ using bilinear form:

$$\langle \alpha, [\theta] \rangle = \int_X \alpha \wedge \theta$$

Re-writing $(*)$, we are close to Riemann-Roch:

From exact sequence

$$h^0(D) := \dim H^0(D) = \underbrace{1}_{\text{from image of } \mathbb{C}} + \dim \ker \underline{A}$$

so $(*)$ can be written:

$$h^0(D) = d - \underbrace{g}_{\dim H^{0,1}} + 1 + \dim \ker \underline{A}^T.$$

$\underline{A}^T : H^{1,0} \to \bigoplus_i T^*X_{p_i}$

[what is this map more precisely?

Want to show $\dim \ker \underline{A}^T$ is $h^0(K-D)$ = dim of space of holom. functions vanishing at D.

There is an obvious map from

$H^{1,0} \to \bigoplus_i T^*X_{p_i}$, the "evaluation map"

$$ev : \alpha \mapsto \bigoplus_i \alpha|_{p_i}$$

and $\ker(ev) = H^0(K-D)$. So we're done if we show ev is const. multiple of $\underline{A}^T$.

Unwinding definitions, given holom. 1-form θ, compute $\int_X A \wedge \theta$

(again restrict to case of single point p for D)

So $A = \bar\partial\beta \cdot \frac{1}{z}$, a $(0,1)$-form
in local coord z centered at p.
$\theta = f(z)\,dz$, a $(1,0)$ form

$$\int_{\substack{\text{small disk}\\ \text{about } 0\\ \text{or } \mathbb{C}}} \underbrace{\bar\partial\beta\, \frac{1}{z} \wedge f(z)\,dz}_{\text{a 2-form}} \underset{\substack{\text{Stokes' Thm}\\ \text{or Cauchy's}\\ \text{thm.}}}{=} \int_{\gamma} \beta\, \frac{1}{z}\, f(z)\,dz$$

since $\operatorname{supp}(\beta)$ small anyway

γ boundary of disk about 0 on which $\beta = 1$.

$$\underset{\text{Res. thm.}}{=} 2\pi i\, f(0).$$

Note that $\bar\partial\beta \cdot \frac{1}{z} f(z)\,dz$ is exact since

$$d(h\cdot\theta) = dh \wedge \theta + h\,d\theta \quad \text{and} \quad d = \partial + \bar\partial,$$

so

$$d(\underbrace{\beta\tfrac{1}{z}}_{h}\ \underbrace{f(z)\,dz}_{\theta}) = \underbrace{(\partial+\bar\partial)(\beta\tfrac{1}{z})}_{\bar\partial(\beta\frac{1}{z})} \wedge f(z)\,dz$$

since ∂ produces dz which, when wedged with other dz gives 0.