Have the anti-commutative ($\partial\bar{\partial} = -\bar{\partial}\partial$) diagram

$$\begin{tikzcd} \Omega^0 \arrow[r, "\bar{\partial}"] \arrow[d, "\partial"'] \arrow[dr, bend right=60, "\frac{i}{2}\cdot\Delta"'] & \Omega^{0,1} \arrow[d, "\partial"] \\ \Omega^{1,0} \arrow[r, "\bar{\partial}"'] & \Omega^2 \end{tikzcd}$$

where $\partial = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) dz$

$\bar{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) d\bar{z}$

$\frac{i}{2}\cdot\Delta$ $\quad$ Δ: Laplacian

Main theorem : X compact R.S. , ρ: 2-form on X , $\exists$ a (unique up to adding constant)

Solution f to $\Delta f = \rho \iff \int_X \rho = 0$.

The main theorem gives relations between de Rham and Dolbeault cohomology.

Remember : $H_X^{0,0} = \ker \bar{\partial}^0$ $\qquad$ $H_X^{0,1} = \operatorname{coker} \bar{\partial}^0 = \Omega^{0,1}/\operatorname{Im}(\bar{\partial})$

$H_X^{1,0} = \ker \bar{\partial}^1$ $\qquad$ $H_X^{1,1} = \operatorname{coker} \bar{\partial}^1 = \Omega^2/\operatorname{Im}(\bar{\partial})$

Corollary of Main Theorem : X compact R.S.

(1) then we have a map of 1-forms $\sigma: \alpha \mapsto \bar{\alpha}$ ($z \mapsto \bar{z}$, $dz \mapsto d\bar{z}$ and vice versa)

which induces an isomorphism $H^{1,0} \xrightarrow{\sim} \overline{H^{0,1}}$

(2) ~~Theorem~~ The bilinear map $B: H^{1,0} \times H^{0,1} \to \mathbb{C}$ given by

$$B(\alpha, [\theta]) = \int_X \alpha \wedge \theta$$

write class here as $H^{0,1}$ defined by coker, so quotient

~~which~~ induces an isomorphism $H^{0,1} \simeq (H^{1,0})^*$

③ There is a natural inclusion map $i : H^{1,0} \longrightarrow H^1$, $\alpha \longmapsto [\alpha]$ ($H^{1,0}$ is holomorphic 1-forms so closed, so map is well defined)

and $$H^{1,0} \oplus H^{0,1} \longrightarrow H^1$$
$$(\alpha, [\theta]) \longmapsto i(\alpha) + \overline{i(\delta^{-1}([\theta])}$$

is an isomorphism.

④ The map $\nu : H^{1,1} \longrightarrow H^2$ induced by inclusion of $\mathrm{Im}(\bar{\partial})$ in $\mathrm{Im}(d')$ is an isomorphism.

pf of ①: To show δ surjective, given any $[\theta] \in H^{0,1}$, want to find f with $\underbrace{\theta' = \theta + \bar{\partial} f}_{\text{i.e. in same class in } H^{0,1}}$ such that $\partial\theta' = 0$. Then $\alpha = \bar{\theta}'$ is a holomorphic 1-form (killed by $\bar{\partial}$) with $[\theta] = -\delta(\alpha)$.

But $\partial\theta' = \partial(\theta + \bar{\partial} f)$ so want f such that

$\underset{\frac{i}{2}\Delta}{\underset{\sim}{\partial\bar{\partial}}} f = -\partial\theta$. By main theorem, such an f exists since $\int_X \partial\theta = 0$ by Stokes' theorem (∂X empty.) and $\partial\theta$ is exact since $d = \partial + \bar{\partial}$ and $\bar{\partial}\theta = 0$ since $\theta \in H^{0,1}$.

For more details, see the proof of Thm 6 in 8.1 of Donaldson's "Riemann Surfaces"

Riemann-Roch formula : Given distinct points $p_1, \dots, p_d$

(weak version) Let $D = \{p_1, \dots, p_d\}$

$H^0(D)$: meromorphic functions on comp. R.S. X with at worst a simple pole at the $p_i \in D$ (and holomorphic elsewhere)

$H^0(K-D)$: holomorphic 1-forms which vanish at each $p_i \in D$.

$h^0(D)$, $h^0(K-D)$ their dimensions, (notation will look weird. Why K? reflecting theory of divisors.)

K: canonical divisor.

then

$$h^0(D) - h^0(K-D) = d - g + 1.$$

(more general versions allow for poles, zeros of arbitrary order. formal sum of points to represent D. Then d in our version is replaced by $\deg(D)$.)

Divisors are covered in Ch. 5 of Miranda.

Sanity check : $X = S^2$.

$H^0(K-D) \subset H^{0,1} = \underset{\text{consts.}}{\cancel{}}$, so $H^0(K-D) = 0$ if $d \geq 1$.

so Riemann-Roch says:

$h^0(D) = d+1$. $\sim$ what are these functions?

Ans: merom. functions on S^2 are rational functions.

e.g. if $\{\infty\} \notin D$; $\dfrac{P(z)}{(z-p_1)\cdots(z-p_d)}$

$\deg. P(z) \leq d$ (dim $d+1$)

(if $> d$, this would create a pole at ∞)