Normal families: Ahlfors reserves term set for set of points,

collection of functions $\mathcal{F}$ consisting of $f : \Omega \to \mathbb{C}$ are called "families"

(or replace $\mathbb{C}$ with favorite metric space S) distance function d

Definition: A normal family is family $\mathcal{F}$ as above
(Ahlfors 5.5)

if every sequence $\{f_n\}$ with $f_n \in \mathcal{F}$ has a subsequence which converges uniformly on every compact subset of Ω.

Note: the limit of the convergent subsequence may not be an element of $\mathcal{F}$.

Compare Bolzano-Weierstrass property: True if and only if metric space is compact.
(for sequences of pts in a metric space)

Plan: find a metric ρ on $\mathcal{F}$ such that convergence in ρ (of a sequence) $\iff$ convergence (of a sequence) uniformly on compact sets in Ω.

then apply Bolzano-Weierstrass thm:

Thm: $\mathcal{F}$ is normal $\iff$ $\overline{\mathcal{F}}$, the closure of $\mathcal{F}$ with respect to ρ, is compact.

(say $\mathcal{F}$ is "relatively compact")

Roughly, it should be hard to be a normal family, since random collection of won't possess "Bolzano-Weierstrass" property.

For the metric ρ, pick an exhaustion of Ω, increasing sequence

$$E_1 \subset E_2 \subset \cdots \subset E_k \subset \cdots \subset \Omega$$

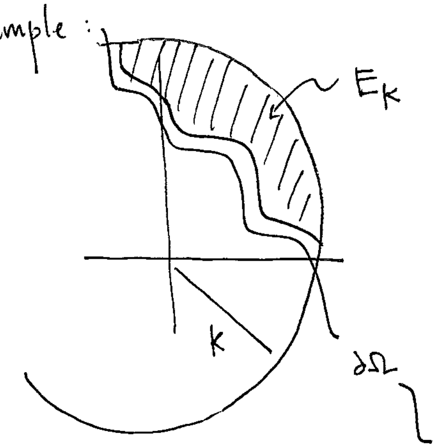
s.t. any compact $E \subseteq \Omega$ belongs to some E_k.

Ahlfors : $E_k : \left\{ z \in \Omega \;\middle|\; |z| \leq k, \quad d(z, \partial\Omega) \geqslant 1/k \right\}$

ugly to draw, clearly compact since closed, bounded.

every compact set in Ω has bounded-below distance from $\partial\Omega$.

Example :



Define

$$\delta_k(f, g) = \sup_{z \in E_k} \underbrace{\frac{d(f(z), g(z))}{1 + d(f(z), g(z))}}$$

$\delta(f,g)$ is itself a distance function and is bounded.

Then set $\rho(f,g) = \sum_{k=1}^{\infty} \delta_k(f,g)\, 2^{-k}$

3 axioms :
$d(x,y) = 0 \iff x = y$
$d(x,y) = d(y,x)$
$d(x,z) \leq d(x,y) + d(y,z)$

are essentially immediate, once you check triangle ineq. for $\delta(f,g)$.

Check that ρ - convergence $\iff$ uniform convergence on compacta.

$(\Rightarrow)$ Suppose $f_n \to f$ in ρ-metric, so given any $\epsilon' > 0$ for $n \gg 0$, $\rho(f_n, f) < \epsilon'$

$\Rightarrow$ (since all terms positive each must be $< \epsilon$) $\delta_k(f_n, f)\, 2^{-k} < \epsilon'$ i.e. $\delta_k(f_n, f) < 2^k \epsilon'$

But then f_n converges to f uniformly in δ metric on E_K:

Given $\epsilon > 0$, pick $\epsilon' = \epsilon \cdot 2^{-k}$, then $\delta_K(f_n, f) < \epsilon$ for $n >> 0$.

If $\delta(f,g) < \epsilon$, then $d(f,g) < \dfrac{\epsilon}{1-\epsilon}$

so also converges in d-metric.

Since f_n converges uniformly to f on all E_K, and every compact $E \subset \Omega$ is contained in some E_K, this completes $(\Rightarrow)$.

$(\Leftarrow)$ If $f_n \to f$ uniformly on every compact set, then for any fixed k, $\delta_K(f_n, f) \to 0$ as $n \to \infty$. And $\underbrace{\sum \delta_K(f_n, f)\, 2^{-k}}_{\rho(f_n, f)} \leq \displaystyle\sum_{k=1}^{\infty} 2^{-k}$ independent of n.

$\Rightarrow$ $\rho(f_n, f) \to 0$ as $n \to \infty$.

(choose n sufficiently large so that $k_1, \dots, k_\ell$ have $\delta_{k_i}(f_n, f)$ as small as want)

So given metric to functions $\mathcal{F}$: $f : \Omega \to \mathbb{C}$, now applying Bolzano-Weierstrass:

$\mathcal{F}$ normal $\Longleftrightarrow$ $\mathcal{F}$ compact (with respect to ρ)
and limits are in $\mathcal{F}$.

remark about how canonical our def'n of ρ was

i.e. $\boxed{\mathcal{F} \text{ normal } \Longleftrightarrow \overline{\mathcal{F}} \text{ compact (with respect to } \rho)}$

($\mathcal{F}$ normal $\Rightarrow$ $\overline{\mathcal{F}}$ normal $\Rightarrow$ $\overline{\mathcal{F}}$ compact)

Is this progress? How to determine if $\overline{\mathcal{F}}$ compact with this funny ρ-metric?

To make use of it, want to exploit alternate characterization of compactness:

<u>Thm</u>: A set is compact $\Leftrightarrow$ complete, totally bounded

(More familiar is its corollary : Subset of $\mathbb{R}/\mathbb{C}$ is compact $\Leftrightarrow$ closed, bounded)

Recall <u>complete</u>: every Cauchy sequence converges

<u>totally bounded</u>: for every $\epsilon > 0$, set can be covered by finitely many balls of radius ϵ.

(3rd characterization: Heine-Borel property that every open cover has a finite subcover is our definition of compact)

$\mathcal{F}$ normal $\Leftrightarrow$ $\bar{\mathcal{F}}$ compact $\Rightarrow$ $\mathcal{F}$ totally bounded ($\exists$ finite set $f_1, \ldots, f_n \in \mathcal{F}$ s.t. any $f \in \mathcal{F}$ has $\rho(f, f_i) < \epsilon$ for some i.)

(Also $\mathbb{C}$ complete $\Rightarrow$ $\bar{\mathcal{F}}$ complete so

$\mathcal{F}$ normal $\Leftrightarrow$ $\mathcal{F}$ totally bounded.)

thus we ~~[illegible]~~ want to characterize when $\mathcal{F}$ totally bounded in terms of original distance d:

<u>Theorem</u> The family $\mathcal{F}$ is totally bounded $\Leftrightarrow$ for every $\epsilon > 0$, E: compact in Ω

$\exists$ $f_1, \ldots, f_n \in \mathcal{F}$ s.t. for every $f \in \mathcal{F}$

$d(f, f_i) < \epsilon$ on E for some i.

pf is application of same proof techniques as above relating convergence in ρ to convergence in δ to convergence in d. (see p. 222 of Ahlfors)