Last week, we computed de Rham cohomology of compact, oriented smooth surfaces – idea: map to $\mathbb{R}$ using path integrals over generators of $\pi_1(S)$, surface integrals over all of S. use Stokes' thm...

On Friday, introduced cx. structure.

realized that $\Omega^1_{X,\mathbb{C}} = \underbrace{\Omega^{1,0}_X}_{\text{cx. linear}} \oplus \underbrace{\Omega^{0,1}_X}_{\text{cx. anti-linear}}$ for choice of local coords x, y on X.

$dz = dx + i\,dy \qquad d\bar{z} = dx - i\,dy$

With

$$\begin{tikzcd} \Omega^0 \arrow[r, "\bar\partial"] \arrow[d, "\partial"'] & \Omega^{0,1} \arrow[d, dashed, "\partial"] \\ \Omega^{1,0} \arrow[r, dashed, "\bar\partial"] & \Omega^2 \end{tikzcd}$$

$df = \partial f + \bar\partial f$

where $\partial f \overset{\text{def}}{=} \dfrac{\partial f}{\partial z}\, dz$

$\bar\partial f \overset{\text{def}}{=} \dfrac{\partial f}{\partial \bar z}\, d\bar z$

(~~Last Friday wrote this in x,y but then arrived here via C-R eqn's~~)

with $\dfrac{\partial f}{\partial z} = \dfrac{1}{2}\left(\dfrac{\partial f}{\partial x} - i\dfrac{\partial f}{\partial y}\right)$

$\dfrac{\partial f}{\partial \bar z} = \dfrac{1}{2}\left(\dfrac{\partial f}{\partial x} + i\dfrac{\partial f}{\partial y}\right)$

If we define

$$\partial(A\, d\bar z) = \frac{\partial A}{\partial z}\, dz \wedge d\bar z$$

$$\bar\partial(B\, dz) = \frac{\partial B}{\partial \bar z}\, d\bar z \wedge dz$$

obtain maps to 2-forms + (anti-comm. diagram $\partial\bar\partial = -\bar\partial\partial$)

so if f holomorphic ($\bar\partial f = 0$)

$df = \partial f = f'(z)\, dz$.

Define ~~[illegible]~~ (1,0)-forms β to be holomorphic 1-form if $\bar\partial\beta = 0$

(i.e. locally expressible as $B\,dz$ with B holomorphic)

Might guess that for cx. analysis only $\bar\partial \circ \partial$ is interesting. $\Omega^{0,1}$ doesn't play a role.

But really need whole diagram.

What is $\bar{\partial} \circ \partial$ in local coords?

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \qquad \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$$

so $$\bar{\partial}\,\partial f = \frac{1}{4}\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) dz \wedge d\bar{z}$$

$$= -\frac{\Delta f}{2i}\, dx \wedge dy$$

Δ : familiar Laplacian whose kernel gave harmonic functions.

(so identifying functions with 2-forms $dx \wedge dy$, recover usual definition of harmonic.

Better : on $\mathbb{R}^2$ this chart valid everywhere, so identification makes sense globally.)

Any theorem about harmonic functions using only local structure of $\mathbb{R}^2$ can be immediately transferred to R.S. setting.

E.g. ① If ϕ real-valued harmonic function on nbhd N of $p \in X$, then $\exists$ open nbhd $\mathcal{U} \subset N$ of p, holom. function f on $\mathcal{U}$ s.t. $\phi = \mathrm{Re}(f)$.

② Maximum principle — ϕ non-const, real-valued harmonic function on connected open set $\mathcal{U} \subseteq X$. Then given $x \in \mathcal{U}$, $\exists\, x' \in \mathcal{U}$ with $\phi(x') > \phi(x)$. (consequence of ① and that holom. maps open)

"Main theorem" : X compact R.S., ρ : 2-form on X

There is a solution f to $\Delta f = \rho \iff \int_X \rho = 0$

Moreover, the solution f is unique up to addition of a constant.

Why is this the "main theorem"? How does it help to classify Riemann surfaces and their ~~holom~~ meromorphic functions?

(Remember, using elementary invariant "degree" of holomorphic map, Prop 4.14 of Miranda: if X is a compact R.S. with merom. function f with a single simple pole, then $X \simeq S^2$.)

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For this, need Dolbeault cohomology.

$$H^{0,0}_X = \ker \bar\partial : \Omega^0 \to \Omega^{0,1} \quad \text{(holomorphic functions)}$$

$$H^{1,0}_X = \ker \bar\partial : \Omega^{1,0} \to \Omega^2 \quad \text{(holomorphic 1-forms)}$$

$$H^{0,1}_X = \operatorname{coker} \bar\partial : \Omega^0 \to \Omega^{0,1} \quad (\text{i.e. } \Omega^{0,1}/\operatorname{Im}(\bar\partial))$$

$$H^{1,1}_X = \operatorname{coker} \bar\partial : \Omega^{1,0} \to \Omega^2 \quad (\text{i.e. } \Omega^2/\operatorname{Im}(\bar\partial))$$

2 claims: $H^{0,1}_X$ is controlling behavior of meromorphic functions on X and is computable using the main theorem.

Explore first claim: When does there exist a meromorphic function on X with lone simple pole at p?

Ans: find local coord. centered at $p \in X$, call it z, consider $1/z$: merom. on nbhd $\mathcal{U}$ of p.

If β: smooth function on X with $\operatorname{supp}(\beta) \subset \mathcal{U}$ and $= 1$ in small nbhd of p (locally constant at p)

then $\beta \cdot 1/z$ is function on $X \setminus p$.

So finding merom. function with pole at p $\iff$ $\exists$ smooth g on X s.t. $g + \beta\, 1/z$ holom. on $X \setminus p$.

Now $\bar\partial\left(\beta \frac{1}{z}\right) = (\bar\partial\beta)\cdot\frac{1}{z}$

1-form with support in $X \setminus \{p\}$ since β loc. const at p.

so we may extend by 0 at p to give $(0,1)$-form on X, call it A.

want holom. function $g + \beta/z$, some holom. g on $X \setminus \{p\}$

i.e. $\bar\partial g = -\bar\partial(\beta/z) = -A$, for given $A \in \Omega^{0,1}_X$

Solution g to:

solution exists $\iff$ $[A] = 0$ in $\operatorname{coker}(\bar\partial^0) = H^{0,1}_X = \Omega^{0,1}_X / \operatorname{Im}(\bar\partial)$

So simplest case: $H^{0,1}_X = 0$, then true.

Generalize to d poles as follows: if ϕ smooth function on $X \setminus \{p\}$ which restricts to merom. function with pole at p in some nbhd of p,

then for some $\lambda \in \mathbb{C}$, $\phi - \lambda \cdot \beta/z$ is (extends to) smooth function on X

$$\Rightarrow \quad [\bar\partial \phi] = \lambda [A] \text{ in } H^{0,1}_X.$$

So given d distinct points $p_1, \dots, p_d$ with $(0,1)$-forms A_i $i = 1, \dots, d$ supported on small annuli about p_i, then we can find meromorphic ϕ with simple poles at $p_1, \dots, p_d$ if $\exists\ \lambda_i$ s.t. $\lambda_1 [A_1] + \dots + \lambda_d [A_d] = 0 \in H^{0,1}_X$.

(Automatic if $\dim(H^{0,1}_X) < d$)

For second claim: Claim map $\delta : H^{1,0}_X \to \overline{H^{0,1}_X}$ given by (induced by) $\alpha \mapsto \bar\alpha$ is isomorphism.

Only hard part is surjectivity: Given $[\theta] \in \overline{H^{0,1}_X}$ find f so that $\theta' = \theta + \bar\partial f$ s.t. $\partial \theta' = 0$ (then $\alpha = \bar\theta'$ is holom. 1-form with $[\theta] = [\delta(\alpha)]$)

i.e. solve: $\partial\bar\partial f = -\partial\theta$. Has sol'n by Main thm + Stokes' thm.