Last week: studying differential forms. Maps:

$$\Omega^0_S \xrightarrow{d} \Omega^1_S \xrightarrow{d} \Omega^2_S$$

functions; 1-forms (maps from S to cotangent space); 2-forms (maps from S to bilinear forms on cotangent space)

$$f \overset{d}{\longmapsto} d(f) = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2$$

quantity appearing in Green's thm.

$$\alpha = \alpha_1 dx_1 + \alpha_2 dx_2 \underset{d}{\longmapsto} \left(\frac{\partial \alpha_2}{\partial x_1} - \frac{\partial \alpha_1}{\partial x_2}\right) dx_1 \wedge dx_2$$

Exploring question, in $\mathbb{R}^2$, of whether $\exists\, f$ s.t. $d(f) = \alpha$ for any 1-form α.

Ans. (Green's thm) Yes, if and only if $d(\alpha) = 0$ "closed form"

In general, for ~~[illegible]~~ surface S, this is more complicated. Its failure is measured by de Rham cohomology. Before discussing it, tackle integration of 2-forms.

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To define integrals, often restrict class of functions or the domain of integration. (for line integrals, continuous functions on rectifiable curves). Here we (initially) restrict to functions on S with compact support — means closure of $\{s \in S \mid f(s) \neq 0\}$ is compact.

S: oriented surface (automatic for Riemann surfaces — holomorphic charts are conformal)

- means that Jacobian (det of matrix of first partials) taken to be positive at all points

If ρ : two-form $= R(x_1,x_2)\, dx_1 \wedge dx_2$ with $\operatorname{supp}(R) \subset \mathcal{U}$: open chart domain,

then $\displaystyle\int_S \rho = \int_{\mathbb{R}^2} R(x_1,x_2)\, dx_1\, dx_2$ ~ Lebesgue integral.

Independent of local chart, even though Lebesgue integration doesn't see orientation, since change of vars is related by Jacobian which is positive.

What if $\operatorname{supp}(\rho)$ is not in single chart? Here $\operatorname{supp}(\rho)$ means write as $R(x_1, x_2)\, dx_1\, dx_2$ in local coords with $R \neq 0$. The fact that $R \neq 0$ is independent of local chart. Use partition of unity — see previous notes...

Fact (General version of Green's/Stokes' thm)

α: compactly supported 1-form on oriented surface S with boundary.

Then: $$\int_{\partial S} \alpha = \int_S d\alpha$$

If ρ doesn't have compact support, can define

$$\int_S \rho = \sup_{\chi} \int_S \chi \rho$$

where χ: any smooth compactly supported function on S taking values in $[0,1]$.

This might be ∞, of course.

Example of partition of unity: K: closed disk of radius $1/2$

$U \subseteq \mathbb{R}^2$: open unit disk

Take $f(x_1, x_2) = F(\sqrt{x_1^2 + x_2^2})$ $F(r)$ any smooth function with $F(r) = 1$ for $r \leq 1/2$

$F(r) = 0$ for $r \geq 3/4$

de Rham cohomology: Given the chain complex

$$\Omega^0_S \xrightarrow{d} \Omega^1_S \xrightarrow{d} \Omega^2_S \quad \text{define}$$

for brevity, can use $d_i : \Omega^i_S \to \Omega^{i+1}_S$

$H^0(S) = \ker(d : \Omega^0_S \to \Omega^1_S)$

$H^1(S) = \ker(d : \Omega^1_S \to \Omega^2_S) / \operatorname{Im}(d : \Omega^0_S \to \Omega^1_S)$

$H^2(S) = \Omega^2_S / \operatorname{Im}(d : \Omega^1_S \to \Omega^2_S)$

Examples: (0) If S connected, $H^0(S) = \mathbb{R}$ (both $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2} \equiv 0$ only if f constant)

(1) If $S = \mathbb{R}^2$, we showed $\ker(d_1) = \operatorname{Im}(d_0)$

so $H^1(\mathbb{R}^2) = 0$.

Also $\Omega^2_S = \operatorname{Im}(d_1)$ since we need to find α_1, α_2 s.t.

$$\frac{\partial \alpha_1}{\partial x_2} - \frac{\partial \alpha_2}{\partial x_1} = R(x_1, x_2)$$

for local coords x_1, x_2 which can serve as global coords in $S = \mathbb{R}^2$.

Take $\alpha_2 = 0$ and $\alpha_1 = \int_0^{x_2} R(x_1, t)\, dt$.

so $H^2(\mathbb{R}^2) = 0$ as well.

(Idea in general: use powerful theorems from integration theory (FTC, Stokes', etc) to easily compute cohomology groups.)

(2) If $S = S^2$, then write $S^2 = U \cup V$ two charts diffeomorphic to $\mathbb{R}^2$

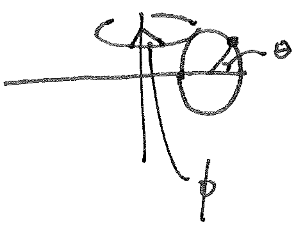
If α: 1-form on S^2 with $d\alpha = 0$, then writing $\alpha|_U$ for its restriction to $U \cong \mathbb{R}^2$, $\exists\, f_U$ s.t. $df_U = \alpha|_U$. Similarly $\exists\, f_V$.

On $U \cap V$, $d(f_U - f_V) = 0 \to f_U - f_V$ constant on $U \cap V$.

We can take this constant, c say, to be zero by changing say $f_U \mapsto f_U - c$ without affecting any of the above claims, regarding df_U. So $f_U = f_V$ on $U \cap V$

Thus the f on S^2 equal to f_u on $\mathcal{U}$, f_v on V, well-defined on all of S^2, with $\alpha = df$. $\Rightarrow$ $H^1(S^2) = 0$.

(3) $S = T$: torus. Coordinatize with pair of angles $\theta, \phi \in [0, 2\pi)$



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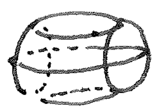

γ_1 : circle with $\theta = 0$, $\phi \in [0, 2\pi)$

γ_2 : circle with $\phi = 0$, $\theta \in [0, 2\pi)$

Consider the map on 1-forms :

$$\alpha \longmapsto \left(\int_{\gamma_1} \alpha \, , \int_{\gamma_2} \alpha \right) \in \mathbb{R}^2 . \qquad (*)$$

Setting $\alpha = d\theta, d\phi$ shows the map is surjective. If f : smooth function on T

then $\displaystyle\int_{\gamma_i} df = 0$.

Thus map $(*)$ induces (surjective) map on $H^1(T) = \ker(d_1) / \operatorname{Im}(d_0)$ to $\mathbb{R}^2$.

claim : map is injective.

pf. of claim : if $\alpha = P\, d\theta + Q\, d\phi$ is closed 1-form such that $\displaystyle\int_{\gamma_2} \alpha = 0$, then for any fixed ϕ, $\displaystyle\int_0^{2\pi} P(\theta, \phi)\, d\theta = 0$ by stokes' thm.

$\Rightarrow$ $f(\theta, \phi) := \displaystyle\int_0^{\theta} P(u, \phi)\, du$ is smooth function on T with $\partial f / \partial \theta = P$

so $\tilde{\alpha} = \alpha - df$ is closed 1-form, which can be written as $\tilde{Q}\, d\phi$ for some $\tilde{Q}$.

But if $\tilde{\alpha}$ closed, then $\tilde{Q}$ constant. So if $\displaystyle\int_{\gamma_1} \tilde{\alpha} = 0$, then $\alpha = df$. //