Last week: Given Riemann surface Y, then holomorphic (proper, non-const.) maps $F: X \to Y$ are in 1-1 correspondence with monodromy rep'ns $\rho: \pi_1(Y \setminus B) \to S_d$ with d: degree. (up to conjugacy)

For converse, we reconstructed R.S. X from $X \setminus F_0^{-1}(B)$ by plugging holes — pasting in disks identified with open set in $X \setminus F_0^{-1}(B)$ at all but origin using $Z \amalg D / \phi$ construction.

Today: Use similar techniques to create compact R.S. from singular algebraic curves. Start with polynomial $P(z,w) = 0$ in $\mathbb{C}^2$.
zero locus

Assume that P is irreducible (so zero locus is connected — a proof we quoted but skipped.)
and that P is not linear polynomial $z - z_0$ for some z_0 (with no dependence on w)

With this assumption, it is possible to show that there are only finitely many points (z,w) with $P = \frac{\partial P}{\partial w} = 0$, and thus only finitely many singular points $P = \frac{\partial P}{\partial w} = \frac{\partial P}{\partial z} = 0$. Call this set Σ.

Note that linear polys in z alone are only irreducible polys in z alone.

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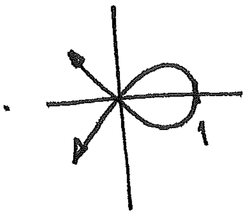
Call zero locus X as usual. Then $X \setminus \Sigma$ is a Riemann surface with local charts given by projection to z or w by implicit function thm (locally like graph)

working example: $w^2 - z^2(1-z) = 0$. Normally $w^2 =$ cubic gives "elliptic curve" — genus 1 R.S. — when the cubic is non-singular.

This has a lone singular point $(0,0)$.

We can't draw surface very easily, but can draw locus of real points:



(remind ourselves that this is just real sol'ns, so surface is connected after removing $(0,0)$)

singularities for which Hessian matrix of second partials

$$\left| \begin{matrix} \partial^2 f/\partial z^2 & \partial^2 f/\partial z \partial w \\ \partial^2 f/\partial w \partial z & \partial^2 f/\partial w^2 \end{matrix} \right| \neq 0 \quad \text{is called a "node"}$$

In our example, Hessian at $(0,0)$: $\left| \begin{matrix} -2 & 0 \\ 0 & 2 \end{matrix} \right|$.

See Miranda III.2 "Nodes of Plane Curve"

This implies our expansion at $(0,0)$ for $P(z,w)$ begins

$$-2z^2 + 2w^2 = 2(w-z)(w+z) + \text{higher order terms}$$

$+$ higher order terms

$$\Rightarrow \quad P(z,w) = (\underbrace{2(w-z) + \cdots}_{Q(z,w)})(\underbrace{(w+z) + \cdots}_{R(z,w)})$$

R, Q have zero loci which are Riemann surfaces.

prove this factorization exists by induction

To resolve the singularity, let F: finite set in $\mathbb{C}$ for which highest degree term in w (a polynomial in z) vanishes

$$\Sigma^+ := \pi_z^{-1}(\pi_z(\Sigma) \cup F) \qquad \pi_z: \text{projection } X \to \mathbb{C}, \ (z,w) \mapsto z$$

finite set since Σ finite, F finite, so $\pi_z(\Sigma) \cup F$ finite

For $z_0 \in \pi_z(\Sigma) \cup F$, $\pi_z^{-1}(z_0) = \{ (z_0, w) \mid P(z_0, w) = 0 \}$ so roots of 1-var. poly in w, provided $P(z_0, w) \not\equiv 0$, which only happens if P divisible by $(z - z_0)$.

Then

$$\pi_z: X \setminus \Sigma^+ \longrightarrow S^2 \setminus \overbrace{\{\pi_z(\Sigma) \cup F \cup \{\infty\}\}}^{E}$$

is a proper holomorphic map.

Now there may still be additional branch points of π_z, so associated monodromy rep'n $\rho : \pi_1(S^2 \setminus \{B \cup E\}) \longrightarrow S_d$ with d: degree of π_z.

By our previous 1-1 correspondence, ρ also defines a compact Riemann surface X^* containing $X \setminus \Sigma^+$ as a dense subset and mapping holomorphically to S^2. (Moreover X^* connected since X connected)

— There is another set that is compact, associated to X, namely the projective curve given by homogenizing original P_1 as subset of $\mathbb{P}^2(\mathbb{C})$.

E.g. $w^2 - z^2(1-z) = 0 \rightsquigarrow w^2 v - z^2 v + z^3 = 0$

$[z : w : v] \in \mathbb{P}^2(\mathbb{C})$.

call it $\overline{X}$.

Proposition: $X \setminus \Sigma^+ \subseteq \overline{X}$ extends to a holomorphic map from $X^* \longrightarrow \mathbb{P}^2(\mathbb{C})$ mapping onto $\overline{X}$.

this isn't R.S., but can define holomorphic map as continuous map holomorphic w.r.t. charts to $\mathbb{C}^2$.

First show that, when forming X^* by gluing in disks, then we can extend

holom. $X \setminus \Sigma^+ \longrightarrow \mathbb{P}^2(\mathbb{C})$

to function meromorphic at origin of disk.

Lemma: p: irred. poly in z, w, n: positive integer.

f: holomorphic function on punctured disk $D \setminus \{0\}$ with $P(z^n, f(z)) = 0$ $\forall z \in D \setminus \{0\}$.

pf of lemma: Since P irreducible, there are only finitely many roots $w_1, \dots, w_N$ s.t. $P(0,w) = 0$. Thus if $|z|$ small, then $f(z)$ must be close to one of w_i. This contradicts property of essential singularities: in every nbhd. of essentially singularity, f is arbitrarily close to any cx. value.

So f is, at worst, meromorphic.

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This implies that for some $m \gg 0$, $z^m \cdot f(z)$ is holomorphic in nbhd. of $z=0$. The map $z \mapsto [z^n, f(z), 1]$ is equal to $z \mapsto [z^{n+m}, z^m f(z), z^m]$ which gives holomorphic map from D to $\mathbb{P}^2(\mathbb{C})$.

and non-vanishing! for particular choice of m

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Back to example: $S^+ : (0,0)$

$$\pi_z : X \setminus \{(0,0)\} \to S^2 \setminus \{0\}, \{\infty\}$$

$$(z,w) \mapsto z$$

deg 2 map. Additional branch point at $z=1$

So monodromy repn $\rho : \pi_1(S^2 \setminus \{0,1,\infty\}) \to S_2$

paste in two disks to $X \setminus \{(0,1), (0,0)\}$ from $(0,0)$, $(0,1)$.

Donaldson: Resulting $X^* \cong \mathbb{P}^1(\mathbb{C})$ with map

$$\mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^2(\mathbb{C})$$

$$[\tau : 1] \to [1 : 1-\tau^2 : \tau - \tau^3]$$

$$[1:0] \to [0:0:1]$$

(via Hurwitz formula?)