If we are being careful about base points, then given base point in Y may not lie in W. So take path from base point y_0 to point y_1 in W, call it α, then apply small loop around b - call it β, then traverse back along α in opposite direction $: \alpha^{-1}$.

Think of α as an identification of fiber of F over y_0 and fiber of F over y_1. If we view the fiber as labelling then different α may give different identifications of labelling. Thus elts. of S_d are only determined up to conjugation, but this preserves cycle type.

Conclusion: Given non-const., proper holom. map $F: X \to Y$, we obtain an integer $\underset{\text{degree}}{(d)}$, a discrete set $\underset{\text{(branch points)}}{B \subset Y}$, and a transitive gp. homom. $\underset{\text{(monodromy repn)}}{\rho: \pi_1(Y \setminus B) \to S_d}$ up to conjugacy.

Thm: Let Y be ~~conn.~~ Riemann surface, B: discrete set in Y. $d \geq 1$ integer, $\rho: \pi_1(Y \setminus B) \to S_d$ transitive gp. hom., then there exists a pair (F, X) with $F: X \to Y$ proper holom. map of Riemann surfaces s.t. its monodromy repn is ρ. Such (F, X) are unique up to equivalence.

Pf. By the theory of covering spaces, if given a subgp. H of index d of $\pi_1(Y \setminus B)$ then we may form a cover $F_0: X_0 \to Y \setminus B$ of degree d.

Pick an index $\in \{1, \ldots, d\}$, say 1, consider $[\gamma] \in \pi_1(Y \setminus B)$ s.t. $\rho([\gamma])(1) = 1$

That is, $[\gamma]$ maps to a permutation fixing 1. These $[\gamma]$ form a subgroup H of index d. Take corresponding cover.

This realizes the monodromy rep'n by our earlier abstract description.

Initially X_0 is just connected topological space. But since Y, and hence $Y \setminus B$, are R.S., then make charts for X_0 via composing covering map with charts for $Y \setminus B$. Requiring that F_0 is holomorphic with respect to R.S. structure on X_0 specifies it uniquely.

Have: $F_0 : \underbrace{X_0}_{\substack{\text{playing role} \\ \text{of } X \setminus F^{-1}(B) \\ \text{from before}}} \longrightarrow Y \setminus B$ Want: $F : X \longrightarrow Y$.

Need to explain how to fill in pts. of $F^{-1}(B)$ over branch points B.

Pick $b \in B$, small disk around b D_0, γ : boundary so that $[\gamma]$ defines conj. class in $\pi_1(Y \setminus B)$. Apply monodromy rep'n, then

$\rho([\gamma])$ = permutation σ cycle type $m_1, \dots, m_k$ s.t. $\sum m_j = d$.

connected components of $F_0^{-1}(D_0 \setminus \{b\})$ correspond to cycles.

Pick one, Z. Then Z is conn. cover of $D_0 \setminus \{b\}$ of degree m_j with generator of $\pi_1(D_0 \setminus \{b\})$ mapping to an m_j-cycle in S_{m_j}

$\Rightarrow$ $Z \simeq D^*$: disk in $\mathbb{C}$ with map $z \mapsto z^{m_j}$

taking us from $\overbrace{Z \to D_0 \xrightarrow[\phi_b]{} D^*}^{\text{map } \phi}$

Consider $X = X_0 \amalg D / \phi$ where D is non-punctured disk in $\mathbb{C}$

So points of $Z \subset X_0$ are identified with pts. of $D^* \subset D$ via

ϕ, and unique point not in X_0 is $\{0\} \in D$.

Know $X_0 \amalg D / \phi$ is R.S. provided we can show it is Hausdorff.

$$\begin{array}{c} X_0 \amalg D/\phi \\ \| \\ X \end{array}$$

i.e. given $u, v \in X$, want disjoint open sets $\mathcal{U}, \mathcal{V}$ containing u, v respectively.

if $u, v \in X_0$, done since X_0 is R.S. so in particular Hausdorff.

only difficulty: $u \in X_0$, $v = 0 \in D$. But $F_0 : X_0 \to Y \setminus B$ maps

$u \mapsto F_0(u) \neq b$ so $\exists$ open nbhd N_u of $F_0(u)$ in $Y \setminus B$

which is disjoint from small open nbhd. (punctured) N_v of b.

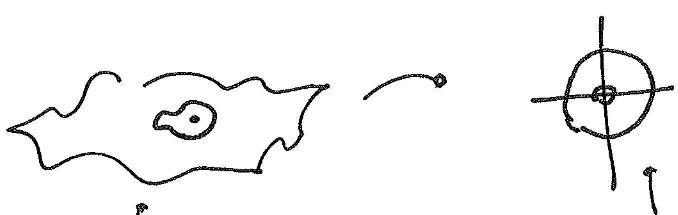
$\Rightarrow$ $F_0^{-1}(N_u)$ and $\{0\} \cup \phi^{-1}(F_0^{-1}(N_v))$ are disjoint in X

[In general can always try construction $X \amalg D / \phi$ where $D^* \underset{\phi}{\overset{\sim}{\leftarrow}} Z$ open set in X

when will result be Hausdorff? If and only if

ϕ extends to holomorphic map ~~[illegible]~~ from X to D.

Picture:



we don't want closure of Z in X to fill middle.

(i.e. if point we were trying to "fill" already existed in X)

closure in $\mathbb{C}$ fills puncture at 0.

]

See p. 66 of Miranda "Plugging holes in Riemann surfaces"

The fact that F_0 extends to a holomorphic map $F : X \to Y \setminus \{B \setminus b\}$

is clear, since locally it is an m^{th} power map, which has unique extension
at b at its center.

This isn't the end, we need to repeat this for each cycle in monodromy repn
ρ of $[\gamma]$ for b, then repeat for each $b \in B$.

Then done, upon checking the result $\overset{F}{\vee}$ is proper. — for example if Y
is compact, then X compact since it is expressible as ~~a~~ union
of finitely many compact sets : $F_0^{-1}(Y - D_0)$ D_0 : open nbhd. of $b \in B$
and closures of open disks from hole charts
for cycles in $\rho[\partial D_0]$.

so $Y \setminus D_0$ compact
so $F_0^{-1}(Y \setminus D_0)$ compact.

Next time : revisit compactifying algebraic curves (e.g. hyperelliptic)
with new understanding of monodromy / plugging holes.
won't assume it is smooth. Just polynomial in two variables in $\mathbb{C}^2$,
irreducible, not linear of form $z - z_0$.

Facts : $X = \{ (z,w) \in \mathbb{C}^2 \mid P(z,w) = 0 \}$ then
P as above

- X is connected.
- There are only finitely many points (z,w) where P, $\frac{\partial P}{\partial w}$ both vanish.

S : singular set $= \{ (z,w) \in X \mid \frac{\partial P}{\partial z} = \frac{\partial P}{\partial w} = 0 \}$ (finite by fact above)

$X \setminus S$ is Riemann surface. Want to compactify this.