On Friday, we were exploring genus of compact Riemann surfaces,

cut out of $\mathbb{P}^2(\mathbb{C})$ by homogeneous polynomial of small degree.

linear $\simeq S^2 \simeq \mathbb{P}^1(\mathbb{C})$

quadratic : $F(x,y,z) = v^T \underbrace{\begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}}_{A_F} v$ with $v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

We had shown X_F : R.S. made from $F \simeq S^2$ if we could

show that $A_F \sim I_3$: 3x3 identity matrix with $A \sim B$ if $\exists\, C$ such that $C^T A C = B$.

(this relation says, since A arbitrary, that all R.S. defined by quad. form are isomorphic. Showed $F(x,y,z) = xz - y^2$ is isom. to S^2.)

Left to show $A = C^T C$ with C invertible.

Error from Friday: Can't use spectral theorem that allows us to diagonalize. For cx. matrices requires $A = \bar{A}^T$. cx-conj.

Can't use $A = \mathcal{U}^T D \mathcal{U}$ with $\mathcal{U}$: upper triangular.

since this requires non-vanishing of minors.

So followed Clemens: if A not 0-matrix, then can find v_1

s.t. $\langle v_1, v_1 \rangle = 1$ where $\langle v_1, v_1 \rangle = v_1^T \cdot A v_1$

Find invertible matrix L s.t. $L(v_1) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and so that

$L^T A L$ has form: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \multicolumn{2}{c}{\multirow{2}{*}{$\begin{pmatrix} 2\times 2 \\ \text{symm.} \end{pmatrix}$}} \\ 0 & & \end{pmatrix}$ $\begin{pmatrix} d' & e' \\ e' & f' \end{pmatrix}$

Repeat. Either $d', e', f' = 0$ or $\exists\, v_2$

with $\langle v_2, v_2 \rangle = 1$, make change of vars. Arrive at C.

Find that A's grouped into equivalence classes by rank. (# of 1's on diagonal)

since A invertible, has full rank.

<u>Fact</u>: Any $\overset{\text{invertible}}{\text{symmetric}}$ A is expressible as $T^T T = A$ for some invertible T.

(A ~~is diagonalizable, write as CDC^{-1} with D diagonal, composed of eigenvalues.~~

~~A symmetric, so we can~~ ~~normalize eigenvectors (i.e. columns of C)~~ ~~then eigenvectors form orthonormal basis~~

~~to have length 1.~~

and $\cancel{C}C^T = I$ so get $A = \underbrace{\mathcal{U}^- D^{1/2}}_{T^T} \underbrace{D^{1/2}\mathcal{U}^T}_{T}$)

If A symmetric, then can write $A = \mathcal{U}^- D \mathcal{U}$
with $\mathcal{U}^-$ lower triangular, $\mathcal{U}$ upper triangular

That is, the Riemann surface for any symmetric matrix A is isomorphic to the identity matrix, that is they are all isomorphic.

Just pick convenient choice of F to study genus.

$F = xz - y^2$ is non-singular. Then have isom. $\mathbb{P}^1 \longrightarrow X_F$

$$[r:s] \longmapsto [r^2 : rs : s^2]$$

Check easily this is holomorphic map.

$$[x:y] \text{ or } [y:z] \longleftarrow [x:y:z]$$

More clever: Reconstruct compact Riemann surface from pair of affine curves using charts.

<u>Prototype</u>: Riemann sphere: obtained by gluing two copies of $\mathbb{C}$

(had two charts: proj. from north / south poles which mapped onto $\mathbb{C}$)

overlap: $\mathbb{C}^\times$ with transition map $\phi(z) = 1/z$.

identify points in overlap according to ϕ.

<u>More generally</u>: X, Y R.S. with open sets $\mathcal{U} \subset X$, $V \subset Y$ and homeom.

$\phi : \mathcal{U} \to V$, then "glue" X and Y by forming

$$Z = \underset{\substack{\uparrow \\ \text{disjoint union}}}{X \amalg Y} \Big/ \phi = \{ x \in X - \mathcal{U},\ y \in Y - V,\ (u, \phi(u)) \text{ for } u \in \mathcal{U} \}$$

topology on Z: quotient topology from $\pi: X \amalg Y \longrightarrow X \sqcup Y / \phi =: Z$

so $\Omega \subset Z$ is open iff $\pi^{-1}(\Omega)$ open in $X \amalg Y$.

(this of course works for X, Y topological spaces, but ...)

We can further put complex structure on Z if $\phi: U \to V$ is isomorphism of Riemann surfaces.

Prop: $U \subset X$, $V \subset Y$ R.S. $\phi: U \to V$ isom.

$\exists!$ cx. structure on $Z = X \amalg Y / \phi$ s.t. inclusions of X, Y into Z are holomorphic maps. (Resulting Z is conn., not nec. Hausdorff. If Hausdorff, then Z is a R.S.)

pf: Let $j_X: X \hookrightarrow Z$, $j_Y: Y \hookrightarrow Z$ be natural inclusions. Given chart $\psi: U_\alpha \to \psi(U_\alpha)$ on X

then $j_X(U_\alpha)$ is open in quotient topology

(the set $U_\alpha = U_\alpha \cap (X - U) \cup \underbrace{U_\alpha \cap U}_{\text{open}}$

so $\pi^{-1}(j_X(U_\alpha)) = U_\alpha \amalg \underbrace{\phi(U_\alpha \cap U)}_{\text{open in } Y}$)

since ϕ holom. so open map

charts: $\psi_X \circ j_X^{-1}$ with ψ_X: chart on X

$\psi_Y \circ j_Y^{-1}$ with ψ_Y: chart on Y

homeomorphisms which cover Z. Check compatibility, which is easy since charts agree only on identified U and V, related by isom. ϕ.

So reduces to compatibility of original charts.

Moreover, each of these maps must be charts if j_X, j_Y are to be holomorphic.

Now use the gluing principle for affine curves.

Consider smooth affine curve given by

$X : \{(x,y) \mid y^2 = h(x)\}$ h has degree $2g+1$ or $2g+2$, distinct roots.

$\mathcal{U} : \{(x,y) \in X \mid x \neq 0\}$

$Y : \{(z,w) \mid w^2 = k(z)\}$ $k(z) := z^{2g+2} h(1/z)$ (poly. in z with distinct roots)

$V : \{(z,w) \in Y \mid z \neq 0\}$

$\phi : \mathcal{U} \to V$ isomorphism

$(x,y) \longmapsto (z,w) := (1/x, y/x^{g+1})$

$(1/z, w/z^{g+1}) \longmapsfrom (z,w)$

claim: $Z := X \amalg Y / \phi$ is compact Riemann surface of genus g.

(remember g appeared in the degree of the affine curves h, k defining X, Y)

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pf. of claim: compactness follows since

Z is union of compact sets (viewed as subsets of Z via inclusion)

$\{(x,y) \in X \mid \|x\| \leq 1\}$ and $\{(z,w) \in Y \mid \|z\| \leq 1\}$.

Calculate genus using Hurwitz formula, as follows: $X = \{(x,y) \mid y^2 = h(x)\}$ has map, proj. to x, which is holomorphic on all of X. Extend this to a map $\pi : Z \to \mathbb{C}_\infty$ (defined at points in $Y \setminus V = \{(0,w) \mid w^2 = k(0)\}$ so that π continuous.)

(holomorphic map as merom. function)

$\deg(\pi) = 2$ since $y^2 = c$ has 2 sol'ns if $c \neq 0$. Branch points are zeros (i.e. roots) of $h(x)$ and ∞ if $\deg(h)$ odd.

Thus, for either case, $2g+1$ or $2g+2$ have $2g+2$ points with multiplicity 2.

these appear in "error" term of Hurwitz formula.

$$-\chi(Z) = \deg(\pi)\cdot(-\chi(\mathbb{C}_\infty)) + \underbrace{\text{error}}_{2g+2}$$

$$\chi(Z) = -4 + 2g+2 = 2g-2 \qquad \text{i.e. genus of } Z \text{ is } g.$$

Meromorphic functions on hyperelliptic Riemann surfaces.

Described similarly to merom. functions on elliptic curve $\mathbb{C}/\Lambda$

There we broke up elliptic functions into even, odd, used $\wp, \wp'$ to describe resulting pieces.

Here introduce similar involution (order 2 automorphism) of Z:

$$\sigma : Z \to Z$$

taking $(x,y) \in X \mapsto (x,-y)$

$(z,w) \in Y \mapsto (z,-w)$

It is a holomorphic map on Z, so given merom. f on Z

then $\sigma^* f := f\circ\sigma$ is merom. on Z.

And $f + \sigma^* f$ is σ^*-invariant, since $\sigma^2 = \text{id}$.

Notice that projection $\pi : Z \to \mathbb{C}_\infty$ commutes with σ: $\pi\circ\sigma = \pi$

so basic example of σ^* invariant function is pullback under π of meromorphic function r (for "rational") on $\mathbb{C}_\infty$.

<u>Lemma</u>: g merom. on Z s.t. $\sigma^* g = g$. Then $\exists!\ r \in \mathbb{C}_\infty$ s.t. $g = r\circ\pi$.