pf of Hurwitz formula :

Choose triangulation of Y so that each branch point is a vertex.

Let v, e, f be the number of vertices, edges, faces (resp.) in this triangulation. Pull the triangulation back to X via F.

Let v', e', f' be the number of vert, edges, faces of the resulting triangulation of F.

(in particular, each vertex among the v' is a ramification point of F, by definition)

open faces contain no branch point, so each face lifts to $\deg(F)$ faces in X. Similarly for open edges.

For vertex $q \in Y$,

$$|F^{-1}(q)| = \deg(F) + \sum_{p \in F^{-1}(q)} (1 - \mathrm{mult}_p(F))$$

$$\text{Thus} \quad v' = \deg(F) \cdot v + \underbrace{\sum_{q \in Y} \sum_{p \in F^{-1}(q)}}_{\sum_{\text{vertices } p \in X}} (1 - \mathrm{mult}_p(F))$$

Putting it all together

$$2g(X) - 2 = -\chi(X) = -v' + e' - f'$$

$$= -\deg(F) \cdot \underbrace{\chi(Y)}_{2-2g(Y)} + \sum_{\text{vertices } p \in X} \mathrm{mult}_p(F) - 1$$

but if these aren't ramified they contribute 0 to sum.

so done.

Examples so far: $S^2 \simeq \mathbb{P}^1(\mathbb{C})$, $\mathbb{C}/M$, M: lattice, smooth affine/projective curves

or cut out Riemann surface from $\mathbb{P}^n$ with suitable collection of equations

(homog. polys. locally like
$n-1$ indep. equations)

Goal: exhibit examples of compact Riemann surface
of arbitrary genus g.

Easiest way to build compact Riemann surface: cut out of $\mathbb{P}^2$ using nonsingular homog. polynomial.

e.g. linear: $F(x,y,z) = ax + by + cz = 0$.

is non-singular (partials never vanish), isomorphic to $\mathbb{P}^1$:

$\mathbb{P}^1 \to X$ if, for example, $a \neq 0$.

$[r:s] \mapsto [-(br+cs)/a : r : s]$

quadratic ("conics"): Write in matrix form:

$$F(x,y,z) = \underbrace{(x,y,z)}_{v^T} \underbrace{\begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}}_{\substack{\parallel \\ A_F}} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{v} = ax^2 + 2bxy + 2cxz + dy^2 + 2eyz + fz^2$$

(symmetric matrix)

F non-singular $\Longleftrightarrow$ A_F invertible. (vector of 3 partials is $2 \cdot A_F \cdot v$)

and null space is $\{0\}$ iff A_F invert.
remember $\{0\}$ is excluded from $\mathbb{P}^2$.

Given A_F, T non-singular

then $T^T A_F T$ is another non-sing. symmetric matrix

call it B_G. with assoc. quad. form G.

If X: R.S. assoc to F, A_F, Y: R.S. assoc. to G, B_G

Then map $Y \to X$ is isomorphism of R.S. Just need to check

$v \mapsto T \cdot v$

$T^{-1}x \leftarrow x$

it is holomorphic, easy since T linear and charts are given by projection

Fact: Any invertible symmetric A is expressible as $T^T T = A$ for some invertible T.

(~~A is diagonalizable, write as CDC^{-1} with D diagonal, composed of eigenvalues.~~
~~A symmetric, so we can~~ [illegible] ~~normalize eigenvectors (i.e. columns of C)~~
~~to have length 1. Then eigenvectors form orthonormal basis~~
~~and $CC^T = I$~~ so get $A = \underbrace{\mathcal{U}^- D^{1/2}}_{T^T} \underbrace{D^{1/2} \mathcal{U}^T}_{T}$)

If A symmetric, then can write $A = \mathcal{U}^- D \mathcal{U}$
with $\mathcal{U}^-$ lower triangular, $\mathcal{U}$ upper triangular

That is, the Riemann surface for any symmetric matrix A is isomorphic to the identity matrix, that is they are all isomorphic.

Just pick convenient choice of F to study genus.

$F = xz - y^2$ is non-singular. Then have isom. $\mathbb{P}^1 \longrightarrow X_F$

$$[r:s] \longmapsto [r^2 : rs : s^2]$$

Check easily this is holomorphic map.

$$[x:y] \longleftarrow [x:y:z]$$

or

$$[y:z]$$

More clever: Reconstruct compact Riemann surface from pair of affine curves using charts.

Prototype: Riemann sphere: obtained by gluing two copies of $\mathbb{C}$

(had two charts: proj. from north / south poles which mapped onto $\mathbb{C}$)

overlap: $\mathbb{C}^\times$ with transition map $\phi(z) = 1/z$.

identify points in overlap according to ϕ.

More generally: X, Y R.S. with open sets $\mathcal{U} \subset X$, $V \subset Y$ and homeom. $\phi: \mathcal{U} \to V$, then "glue" X and Y by forming

$$Z = X \amalg Y / \phi = \{ x \in X - \mathcal{U},\ y \in Y - V,\ (u, \phi(u)) \text{ for } u \in \mathcal{U} \}$$

disjoint union