Last lecture, we used global invariant "degree of F" $F: X \to Y$, compact

$$\deg(F) := \sum_{p \in F^{-1}(y)} \mathrm{mult}_p(F) \quad (\text{indep. of } y \in Y)$$

Alternately, can assume F proper: For any compact set $K \subset Y$ the preimage $F^{-1}(K)$ is compact. (now not assuming X, Y nec. compact)

then used this to show that f merom. on X compact,

$$\sum_p \mathrm{ord}_p(f) = 0.$$

Proposition: f: non-const merom. function on X: compact. Then

$$\sum_p \mathrm{ord}_p(f) = 0.$$

Pf: Let F be the associated holom. map $X \to \mathbb{C}_\infty$

$\{z_i\}$: pts. of X mapping to 0 (zeros)

$\{p_j\}$: pts. of X mapping to ∞. (poles of f)

$$\deg(F) = \sum_i \underbrace{\mathrm{mult}_{z_i}(F)}_{\mathrm{ord}_{z_i}(f)} = \sum_j \underbrace{\mathrm{mult}_{p_j}(F)}_{-\mathrm{ord}_{p_j}(f)}$$

But
$$\sum_p \mathrm{ord}_p(F) = \sum_i \mathrm{ord}_{z_i}(f) + \sum_j \mathrm{ord}_{p_j}(f)$$
$$= \deg(F) - \deg(F) = 0. \quad //$$

Previously asserted this for Riemann sphere using that all ~~...~~ merom. functions were rational functions. (i.e. used strong characterization)

Next: relate degree to topology through genus. Discuss topology basics for a bit.

Often use language of simplicial complexes:

simplices:  ... (has general coord. definition $\sum_{i=1}^{n+1} x_i = 1,\ x_i \geq 0$)

0 1 2 3

simplicial complex: collection of simplices glued so that intersection of any two σ_1, σ_2 is a face of both σ_1, σ_2

e.g. Hatcher Ch. 2.

Euler using them to construct polyhedra, later used to study manifolds via homeomorphisms from simplices to X with compatibility properties.

For us, study 2-dim'l orientable real C^∞-manifolds.

So use only simplices of dim ≤ 2. Given such X, a triangulation is decomposition into closed subsets homeomorphic to 2-simplices (triangles) whose intersections are "faces": simplices of smaller dimension.

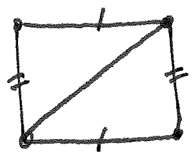
Then for any (finite) simplicial complex, S, assign Euler characteristic χ

$$\chi(S) = (\#\text{ vertices}) - (\#\text{ edges}) + (\#\text{ faces})$$

What is this counting? Euler noticed ~~[illegible]~~ that for regular polyhedra, P,

$$\chi(P) = 2.$$

S:



then $\chi(S) = 1 - 3 + 2 = 0$.

tick marks mean identify corresponding edges

matters how we identify edges (with twist or without?)

torus, $\mathbb{RP}^2$, Klein bottle.

not orientable, so not admitting Riemann surface structure...

<u>Fact</u>: (closed) 2-manifolds have triangulations.
"surfaces"

iterative constructions based on embeddings into $\mathbb{R}^N$, $N \gg 0$, or coverings by balls.

proof is too painful to develop machinery for. Even encyclopedic Hatcher skips it.

<u>Proposition</u>: Let $\chi(X) := \chi(T)$ where T is any triangulation of the orientable, compact 2-manifold X.

Then χ is well-defined (i.e. independent of triangulation) and $= 2 - 2g$ where g: genus.

Could regard this as definition of genus, with $g \geq 0$.

<u>OR</u>: g: maximal # of simple closed curves on X which may be removed s.t. resulting manifold is still connected.

Amazing result connecting degree and genus for holomorphic map.

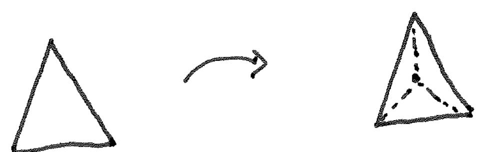
Hurwitz Formula : $F : X \to Y$ non-const. holom. map. , X, Y compact.

Then
$$\underset{-2g(X)+2}{\chi(X)} = \deg(F) \cdot \underset{-2g(Y)+2}{\chi(Y)} + \sum_{p \in X} \underbrace{\mathrm{mult}_p(F) - 1}_{\text{error term arising at ramification points.}}$$

—

prove this after sketching pf. of previous proposition:

A refinement of a triangulation is result of combining elementary refinements:

① 

+ 1 vertex
+ 3 edges
+ 2 faces

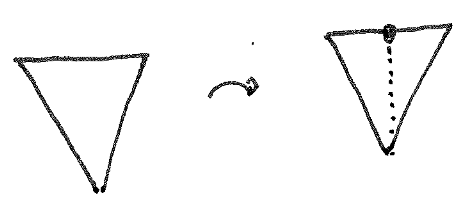
Add vertex to interior of 2-simplex

A refinement of ~~an~~ triangulation is the triangulation resulting from applications of these moves.

② 

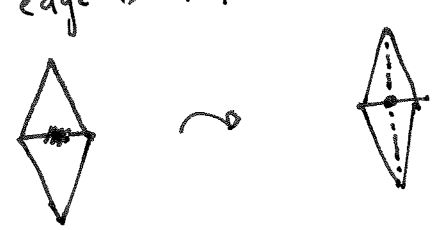
Add vertex to interior of 1-simplex

if manifold has boundary, and edge is boundary edge:

2a 

+ 1 vertex
+ 2 edges
+ 1 face

if edge is interior

2b 

+ 1 vertex
+ 3 edges
+ 2 faces

Key points :

I. Euler characteristic is invariant under elementary refinement, and hence under refinement.

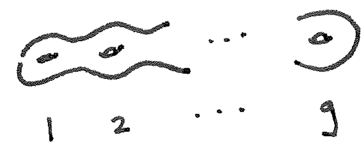
II. A pair of triangulations have a common refinement.

To show II., superimpose two triangulations.

<u>claim</u>: add vertices and edges to make the resulting superposition into a triangulation.

This gives independence of $\chi(X)$ with respect to triangulation.

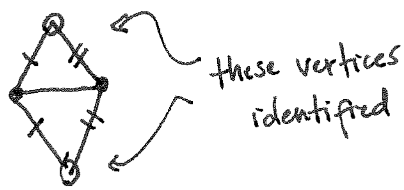
To calculate it in any case, recall that any 2-dim'l orientable real manifold is topologically isomorphic g-holed torus:
(i.e. $\exists$ homeomorphism to...)



"connect sum" of g copies of torus.

So just need to calculate $\chi(X)$ for each g.

e.g. $g=0$



3 v
3 e
2 f

$\chi(S^2) = 2$.

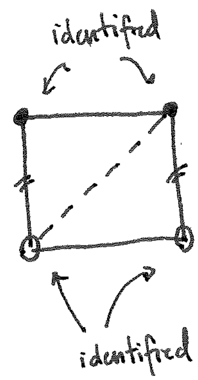
For general g, do "surgery" – remove two disks, attach cylinder.

disk (with boundary):

4 v
5 e
2 f

$\chi(D) = 1$

cylinder



2 v
4 e
2 f

$\chi(\text{Cyl.}) = 0$

So Euler characteristic of above surgery drops characteristic by 2.