Recap: We've been studying Riemann surfaces

collection of $\underbrace{\text{compatible}}_{\text{holomorphic}}$ $\underbrace{\text{local topological isomorphisms to} \cancel{\ } \text{ open sets of } \mathbb{C}}_{\text{"charts"}}$

~~[illegible]~~ study holomorphic functions / meromorphic functions

$$\begin{tikzcd} F: X \arrow[r] & \mathbb{C} \\ \cup & \\ \mathcal{U} \arrow[d, "\phi"'] & \\ V \arrow[uuur, dashed] & \end{tikzcd}$$

$\phi^{-1} \circ F$: holomorphic

Last week: realized these are special cases of "holomorphic maps" of Riemann surfaces.

$$\begin{tikzcd} X \arrow[r, "F"] & Y \\ \cup & \cup \\ \mathcal{U}_X \arrow[d, "\phi_X"'] \arrow[r, bend left] & \mathcal{U}_Y \arrow[d, "\phi_Y"] \\ V_X & V_Y \end{tikzcd}$$

$\phi_Y \circ F \circ \phi_X^{-1}$: holomorphic from $\mathbb{C} \to \mathbb{C}$

($Y = \mathbb{C}$: holomorphic function on X

$Y = \mathbb{C}_\infty$: merom. function on X)

local theory is same as that for ~~[illegible]~~ cx. ~~[illegible]~~ holomorphic functions

- open mapping thm
- identity thm (agreeing on limit pt)
- discreteness of preimages under F: holom.

On Friday, began to study invariants.

multiplicity of F at $p \in X$:

order of vanishing of ~~[illegible]~~ Taylor series

at $\phi_X(p) =: z_0$ in coord $(z - z_0)$.

as expansion ~~[illegible]~~ for $(w - w_0)$ (with $z_0 \mapsto w_0$ under $\phi_Y \circ F \circ \phi_X^{-1}$)

multiplicity is "generally" equal to 1. ($\{p : \mathrm{mult}_p(F) > 1\}$ is discrete in X)

e.g. $z \mapsto z^3$ is holom. map. mult. $= 3$ at $z = 0$ and $= 1$ elsewhere.

b/c mult. at $z = 5$: $\quad w - \underbrace{w_0}_{5^3} = \sum_{n=1}^{3} a_n (z-5)^n \quad a_n = \frac{f^{(n)}(5)}{n!}$ in part., $f^{(1)}(5) \neq 0$

example 2 : f: meromorphic function on X, then let F be associated holomorphic map :

$$F(p) = \begin{cases} f(p) & \text{if } p \text{ not pole} \\ \{\infty\} & \text{if } p \text{ pole} \end{cases}$$

if p not a pole, then $\operatorname{mult}_p(F) = \operatorname{ord}_p(f - f(p))$

in particular if p is a zero, $\operatorname{mult}_p(F) = \operatorname{ord}_p(f)$.

if p is a pole, use chart from stereo. proj. from south pole. Get $\operatorname{mult}_p(F) = -\operatorname{ord}_p(f)$.

or map $\{\infty\} \to \{0\}$ equivalently.

(think in terms of chart centered at origin)

Use the local invariant to make global one - degree(F), $F: X \to Y$ holom. X, Y compact

Given any $y \in Y$, consider $\sum_{p \in F^{-1}(y)} \operatorname{mult}_p(F)$.

If X, Y compact, then $F^{-1}(y)$ finite set, so sum is well defined.

Proposition : This sum is a fixed constant, independent of $y \in Y$. (called $\deg(F)$)

pf : show that $y \mapsto \sum_{p \in F^{-1}(y)} \operatorname{mult}_p(F)$ is locally const. function.

i.e. for every $y \in Y$ $\exists$ nbhd on which function is constant

Since Y connected, then must be constant function.

Lemma : if $F^{-1}(y) = \{x_1, \ldots, x_n\} \in X$, then

if y' near y, $F^{-1}(y')$ contained in nbhds of x_i.

pf of Lemma : if $\exists$ y' arbitrarily close to y whose preimages under F are not all contained in nbhds of x_i. Then construct sequence of x''s outside nbhds of x_i whose images under F converge to y.

Since X compact, can extract a convergent subsequence $\{p_n\}$ in X

with $p_n \to x \in X$, some x, $\lim_n F(p_n) = y$. But since

F continuous, must have $F(x) = y$. But this is a contradiction since

then $x \in \{x_1, \dots, x_n\}$ and is limit pt of p_n's which lie outside all nbhds of x_i //

So to analyze whether our sum is locally constant, we can use

charts for fixed y and $F^{-1}(y) = \{x_1, \dots, x_n\}$.

Here we've seen that we can pick nice charts: centered at x_i and at y.

and of form $w = z_i^{m_i}$ (z_i : local coord for x_i)

m_i : some integer ≥ 1.

But each of $z_i \mapsto z_i^{m_i}$ has property that $\deg(z \mapsto z^{m_i})$ is locally const. function taking value m_i.

so in total $\deg(F) = \sum_{x_i} m_i$,

and we're done. //

Corollary: X: compact Riemann surface, f: meromorphic on X with single simple pole, then $X \simeq \mathbb{C}_\infty$ as Riemann surfaces.

Pf: single simple pole $\Rightarrow$ if $F : X \to \mathbb{C}_\infty$ corresponds to f,

then $\deg(F) = -\operatorname{ord}_p(f)$ where p = pole.

$= 1$.

But degree one map is 1-1, so must have isomorphism

(earlier we proved non-const holom. map $F : X \to Y$ is onto if X compact)

Proposition: f: non-const merom. function on X: compact. Then

$$\sum_p \mathrm{ord}_p(f) = 0.$$

Pf: Let F be the associated holom. map $X \to \mathbb{C}_\infty$

$\{z_i\}$: pts. of X mapping to 0 (zeros)

$\{p_j\}$: pts. of X mapping to ∞. (poles of f)

$$\deg(F) = \sum_i \underbrace{\mathrm{mult}_{z_i}(F)}_{= \mathrm{ord}_{z_i}(f)} = \sum_j \underbrace{\mathrm{mult}_{p_j}(F)}_{= -\mathrm{ord}_{p_j}(f)}$$

But
$$\sum_p \mathrm{ord}_p(F) = \sum_i \mathrm{ord}_{z_i}(f) + \sum_j \mathrm{ord}_{p_j}(f)$$

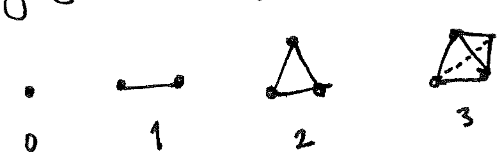
$$= \deg(F) - \deg(F) = 0. \quad //$$

Previously asserted this for Riemann sphere using that all merom. functions were rational functions. (i.e. used strong characterization)

Next: relate degree to topology through genus. Discuss topology basics for a bit.

Often use language of simplicial complexes:

simplices:



... (has general coord. definition $\sum_{i=1}^{n+1} x_i = 1$, $x_i \geq 0$)

simplicial complex: collection of simplices glued so that intersection of any two σ_1, σ_2 is a face of both σ_1, σ_2

e.g. Hatcher Ch. 2.

Euler using them to construct polyhedra, later used to study manifolds via homeomorphisms from simplices to X with compatibility properties.