Last time, proved analogues of thms about isolated zeros and max. mod. principle for holomorphic functions on Riemann surfaces. An immediate corollary was that there are <u>no</u> non-constant holom. functions on compact R.S.

(here we mean holomorphic on the <u>entire</u> Riemann surface)

What about meromorphic functions?

Now polynomials are examples on S^2, where if $\deg(f) \geq 1$ then f has a pole at $\{\infty\}$.

$\{\infty\} = (0,0,1)$ in $S^2 \subseteq \mathbb{R}^3$

Moreover rational functions are meromorphic on S^2, where $\{\infty\}$ will be zero or pole (or neither) for f/g depending on $\deg(f) - \deg(g)$.

Others? What about $\mathbb{P}^1(\mathbb{C})$, projective line?

Now coords are $[z_1 : z_2]/\sim$ so can take $f(z_1, z_2)$: homog. poly in two vars.

Is this holom. function on X? No, since $f(\lambda z_1, \lambda z_2) = \lambda^d f(z_1, z_2)$

$\Rightarrow$ need $d = 0$.

But can take rational functions f/g with f, g homog. of same degree. Then

$$\frac{f(\lambda z_1, \lambda z_2)}{g(\lambda z_1, \lambda z_2)} = \frac{f(z_1, z_2)}{g(z_1, z_2)}$$

so well-defined on X and gives meromorphic function with poles at zeros of g. (Note: defines holom. function on nbhds avoiding zeros of g)

Try to describe all meromorphic functions (globally defined) on these two Riemann surfaces S^2, $\mathbb{P}^1(\mathbb{C})$.

E.g. for S^2, compact so given f meromorphic has finitely many zeros, poles. Construct a rational function r with matching zeros/poles of given orders in finite plane.

If write $S^2 = \{ (x,y,w) \mid x^2+y^2+w^2=1 \}$ identify

$\mathbb{C}$ with points $\neq (0,0,1)$. Call z local coord. for chart.
values in $\mathbb{C}$

Consider $g = f/r$: a merom. function on $\mathbb{C}_\infty = S^2$ with

no zeros/poles in finite plane, and has Taylor series expansion in local

coord z (say about the origin) valid for all $z \in \mathbb{C}$:

$$g(z) = \sum_{n=0}^{\infty} c_n z^n .$$

In a punctured nbhd of ∞, other local coordinate $w = 1/z$, we

have $g(w) = \sum_{n=0}^{\infty} c_n w^{-n}$. But know g is merom. at
north pole $(w=0)$

$\Rightarrow$ only finitely many of c_n are non-zero.

i.e. g is polynomial in z. (with no zeros in $\mathbb{C}$)

hence g constant $\Rightarrow$ f is a rational function.

Not surprisingly, pf is similar for $\mathbb{P}^1(\mathbb{C})$ (we intend to show Friday that $\mathbb{P}^1(\mathbb{C}) \simeq S^2$ as Riemann surfaces)

Now use rational function (for $[z:w]$)

$$r(z,w) = \prod_i (b_i z - a_i w)^{e_i} \quad \text{with} \quad \text{ord}_{[a_i:b_i]}(r) = e_i$$

What about torus? Take different approach from Weierstrass $\wp$-function: theta functions

$$\theta(z) = \sum_{n=-\infty}^{\infty} e^{\pi i [n^2 \tau + 2nz]} \quad \text{for } \tau \in \text{upper half plane } \mathcal{H}$$
(i.e. $\text{Im}(\tau) > 0$)

Create function which is periodic with respect to $\mathbb{Z} + \mathbb{Z}\tau$.

What about smooth curves?

For affine plane curves $X = \{ (z,w) \mid f(z,w) = 0 \}$ non-singular polynomial f.

claim: projection to z defines holom. function on all of X (same for projection to w)

pf Affine plane curve has two charts: proj. to z, proj. to w.

~~[illegible]~~ according to whether $\frac{\partial f}{\partial w} \neq 0$ or $\frac{\partial f}{\partial z} \neq 0$ resp. (implicit function thm.)

Call them π_z, π_w. Analyze

$\underbrace{\pi_z \circ \pi_z^{-1}}_{\substack{\text{trivially holomorphic:} \\ z \mapsto z}}$ and $\pi_z \circ \pi_w^{-1} = \pi_z(g(w), w) = g(w)$ with g holom.

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Corollary: Any polynomial $g(z,w)$, restricted to the set $X = \{ (z,w) \mid f(z,w) = 0 \}$ is holomorphic on X

pf: use that sums and prods of holom. functions are holomorphic, together with above claim, giving z, w holomorphic on X.

Corollary 2: Any rational function $g(z,w)/h(z,w)$ restricted to X is meromorphic unless $h \equiv 0$ (on X).

Note this can happen if $f(z,w)$ defining X divides h. Then $h \equiv 0$ on X.

Nullstellensatz: Suppose f irreducible polynomial and $h = 0 \Rightarrow f = 0$ (i.e. f vanishes at every pt. where h vanishes)

Then f divides h.