Last time, we defined a Riemann surface as 1-dim'l complex manifold meaning we had a system of charts $\{\phi_\alpha : \mathcal{U}_\alpha \to V_\alpha \subseteq \mathbb{C}\}$

with $\bigcup_\alpha \mathcal{U}_\alpha = X$

ϕ_α : homeomorphisms

compatible in complex sense:

$\phi_\beta \circ \phi_\alpha^{-1}$ holomorphic

Also forget the cx. structure, viewing $Re(\phi_\alpha)$, $Im(\phi_\alpha)$ as chart to $\mathbb{R}^2$,

gives a 2-dim'l, real, orientable C^∞-manifold.

Compact ones classified by genus. (up to diffeomorphism)

Don't yet have a notion of isomorphism of Riemann surfaces, postpone this to next week – but it's stronger than being diffeomorphic as C^∞-manifolds. We will see examples of compact Riemann surfaces with same genus but which are not isomorphic.

Examples so far started with topology on X, then formed charts and verified they were homeomorphisms. But need not start with topology in mind.

Can start with X, countable collection $\{U_\alpha\}$ covering X

Declare a subset U of U_α to be open if $\phi_\alpha(U) \subset V_\alpha \subset \mathbb{C}$ is open

Then obtain a topology on X by setting U to be open (for now arbitrary U) iff $U \cap U_\alpha$ open in U_α for all α.

Finally, ϕ_α will be a chart, by definition, if each U_α is open in X and ϕ_α is bijective. U_α is open, by preceding discussion, if $U_\alpha \cap U_\beta$ open in U_α $\forall \beta$.

<u>Plan</u>: Given X, find countable cover $\{U_\alpha\}$

- with bijections $\phi_\alpha : U_\alpha \to$ some open subset V_α of $\mathbb{C}$
- check $\phi_\alpha(U_\alpha \cap U_\beta)$ is open in V_α
- check charts compatible
- check X connected, Hausdorff.

Execute <u>this plan</u> in examples: $\mathbb{C}^2 \setminus (0,0) / \sim$ where $(z,w) \sim (\lambda z, \lambda w)$ for $\lambda \in \mathbb{C}^\times$

geometrically, this is lines in $\mathbb{C}^2$ through origin. i.e. 1-dim'l subspaces of $\mathbb{C}^2$.

Write coordinates as $[z:w]$. Colon meant to suggest ratio, So for $\lambda \in \mathbb{C}^\times$ $[z:w] = [\lambda z : \lambda w]$

"complex projective space" denoted $\mathbb{CP}^1$.

Let's show it has the structure of a Riemann surface

$\mathbb{CP}^1$ is isomorphic to Riemann sphere : think $[z:1]$ as the cx. plane and $[1:0]$ as the point at infinity.

so strategy should look familiar :

$\mathcal{U}_0 = \{ [z:w] \mid z \neq 0 \}$ $\quad \phi_0 [z:w] = w/z$

$\mathcal{U}_1 = \{ [z:w] \mid w \neq 0 \}$ $\quad \phi_1 [z:w] = z/w$

clearly bijections with $\mathcal{U}_0 \cup \mathcal{U}_1 = X$

$\phi_i (\mathcal{U}_0 \cap \mathcal{U}_1) = \mathbb{C}^\times$, open in $\mathbb{C}$ so ϕ_i's are charts.

$\phi_1 \circ \phi_0^{-1} : s \longmapsto 1/s$ so compatible.

- $\mathbb{CP}^1$ is connected since $\mathcal{U}_0, \mathcal{U}_1$ connected , $\mathcal{U}_0 \cap \mathcal{U}_1 \neq \phi$. *
- $\mathbb{CP}^1$ is Hausdorff since $\mathcal{U}_i$ Hausdorff , so suffices to check if $p \in \mathcal{U}_0 - \mathcal{U}_1$, $q \in \mathcal{U}_1 - \mathcal{U}_0$. But then $p = [1:0]$, $q = [0:1]$ show they are separated by $\phi_0^{-1}(D)$ and $\phi_1^{-1}(D)$ where D: open unit disk

* : connectedness preserved under continuous maps.

(will show next week that $\mathbb{CP}^1$ and S^2 are isomorphic as Riemann surfaces)

<u>Example 2</u> : $M = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$: lattice in $\mathbb{C}$ (so ω_1/ω_2 not real)

$X = \mathbb{C}/M$. Then natural projection map $\pi_M : \mathbb{C} \to X$

Define topology on X by declaring $\mathcal{U}$ open if $\pi_M^{-1}(\mathcal{U})$ open in $\mathbb{C}$.

This definition ensures π_M continuous $\Rightarrow$ since $\mathbb{C}$ connected , X is connected.

π_M is open since if $V \subseteq \mathbb{C}$ open , $\pi_M^{-1}(\pi_M(V)) = \bigcup_{\omega \in M} (\omega + V)$ is open

To form charts, note that M discrete so choose ϵ so that

$2\epsilon < |\omega|$ for every $\omega \in M$. Then an open ball of radius ϵ

about any $z_0 \in \mathbb{C}$, $B(z_0,\epsilon)$, contains no equivalent points mod M.

So $\pi|_{B(z_0,\epsilon)} : B(z_0,\epsilon) \to \pi(B(z_0,\epsilon))$ continuous, open (since π is) and onto by defn.

one-one since ϵ chosen small enough.

charts will be inverses to $\pi|_{B(z_0,\epsilon)}$,

call them $\phi_{z_0} : \pi(B(z_0,\epsilon)) \to B(z_0,\epsilon)$

To check compatibility, given ϕ_{z_1}, ϕ_{z_2} let $T = \phi_{z_2} \circ \phi_{z_1}^{-1}$

on $\phi_{z_1}(\underbrace{\pi(B(z_1,\epsilon)) \cap \pi(B(z_2,\epsilon))}_{\mathcal{U}})$

$T(z) = \phi_{z_2}(\pi(z))$ so $\pi(T(z)) = \pi(z)$

$\Rightarrow$ $T(z) - z \in M$ for all $z \in \phi_{z_1}(\mathcal{U})$

$T(z) - z$ continuous, M discrete $\Rightarrow$ $T(z) - z$ constant on conn. components of $\phi_{z_1}(\mathcal{U})$.

i.e. locally $T(z) = z + \omega$, which is holomorphic.

<u>Example 3</u>: $\Omega \subseteq \mathbb{C}$ region. g analytic on Ω.

$X =$ graph of g $= \{(z, g(z)) \subseteq \mathbb{C}^2 \mid z \in \Omega\}$

X inherits subspace topology from $\mathbb{C}^2$. π: proj. to first component:

$$X \to \mathbb{C}$$
$$(z, g(z)) \mapsto z$$

This makes X into Riemann surface with <u>one</u> chart.

Ex. 4 : smooth affine plane curves.

affine plane curve is zero locus of polynomial $f(z,w)$, $(z,w) \in \mathbb{C}^2$

know from implicit function thm that if $p = (z_0, w_0)$ is zero of f

with $\frac{\partial f}{\partial w}(p) \neq 0$ then $\exists$ nbhd of z_0, and holom. function $g(z)$ on nbhd

s.t. $w = g(z)$ with $g'(z) = -\partial f/\partial z \,/\, \partial f/\partial w$. Roles of z, w in $f(z,w)$ reversible so...

$f(z,w)$ is non-singular at $p = (z_0, w_0)$ if
one of $\frac{\partial f}{\partial w}(p)$ or $\frac{\partial f}{\partial z}(p) \neq 0$. (then locally $f(z,w) = 0$ is a graph)

$X = \{ (z,w) : f(z,w) = 0 \}$ is <u>smooth</u> if non-singular at all points.

Now use previous example to conclude X is Riemann surface,
where charts are either projection onto z or w

one thing to check : Are proj. to z and proj to w compatible?

$\pi_z : U_z \to V_z \subseteq \mathbb{C}$ $\qquad U = U_z \cap U_w$.

$\pi_w : U_w \to V_w \subseteq \mathbb{C}$ $\qquad \pi_w \circ \pi_z^{-1} : \pi_z(U) \to \pi_w(U)$

$\pi_z^{-1} : \underset{\underset{U}{\cap}}{z_0} \mapsto (z_0, g(z_0))$, then $\pi_w \circ \pi_z^{-1}(z_0) = g(z_0)$
which is holom.

last thing to check : X connected. Not true without additional assumption. Miranda's non-example : $f(z,w) = (z+w)(z+w-1)$
whose zero locus is a pair of parallel lines.