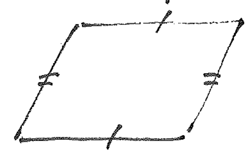
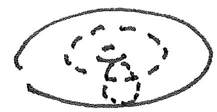
Riemann <u>surfaces</u> : topological spaces which locally look like $\mathbb{C}$.

why <u>study</u> : • arise naturally in problems of analytic continuation – functions weren't single-valued on $\mathbb{C}$ but could be made single-valued on multiple-sheeted covers of $\mathbb{C}$.

• elliptic functions were functions on $\mathbb{C}/M$: fundamental parallelogram



with opposite edges identified.



: complex torus.

• often useful to consider $\mathbb{C} \cup \{\infty\}$ in formulating results about entire functions, e.g. orders of zeros/poles of rational function/merom.-function on $\mathbb{C} \cup \{\infty\}$.

"Riemann sphere"

Today – define Riemann surface carefully, explain why there are examples.

X top. space, $\mathcal{U} \subset X$ open. $\phi : \mathcal{U} \to V \subseteq \mathbb{C}$

homeomorphism : continuous map with continuous inverse

"complex chart"

continuous here : inv. image of open set is open

whose image $\phi(u) = z$ defines complex coordinate

Examples : ① $\mathbb{R}^2 = X$ w/ usual Euclidean topology $\phi(x,y) = x + iy$ is cx. chart for any open set $\mathcal{U} \subset \mathbb{R}^2$.

Many other choices:

$$\phi(x,y) = \frac{x}{1+\sqrt{x^2+y^2}} + i\,\frac{y}{1+\sqrt{x^2+y^2}}$$

is defined for any open set $\mathcal{U}$ in $\mathbb{R}^2$ as well.

② Can always restrict a chart on $\mathcal{U}$ to smaller open set $\mathcal{U}' \subseteq \mathcal{U}$.

③ Can compose with analytic bijection $\psi : V \to W$, $W \subset \mathbb{C}$, then $\phi \circ \psi$ is chart.

(4) Riemann sphere, realized as subset of $\mathbb{R}^3$: $\{(x,y,w) \mid x^2+y^2+w^2=1\} =: S_2$

Think of $\mathbb{C}$ as the plane in $\mathbb{R}^3$ with $w=0$ and $(x,y,0) \longleftrightarrow x+iy = z$.

Last semester, discussing stereographic projection from north pole $(0,0,1)$:

$$\phi_1(x,y,w) = \frac{x}{1-w} + i\,\frac{y}{1-w} \quad (\text{defined on } S_2 \setminus (0,0,1))$$

with inverse $\phi_1^{-1}(z) = \left(\frac{2\,\mathrm{Re}(z)}{|z|^2+1}, \frac{2\,\mathrm{Im}(z)}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1} \right)$

Could have equally well done projection from south pole $(0,0,-1)$:

$$\phi_2(x,y,w) = \frac{x}{1+w} - i\,\frac{y}{1+w}$$ with similar formula for ϕ_2^{-1}.

(defined on $S_2 \setminus (0,0,-1)$)

To define Riemann surface, wanted system of charts such that they cover the space X and,

whenever $\mathcal{U}_1 \cap \mathcal{U}_2 \neq \phi$, want charts to be compatible in the following sense:

if $\phi_1 : \mathcal{U}_1 \to V_1$, $\phi_2 : \mathcal{U}_2 \to V_2$ cx. charts, want

$$\phi_2 \circ \phi_1^{-1} : \underset{\displaystyle \mathbb{C}}{\overset{\phi_1(\mathcal{U}_1 \cap \mathcal{U}_2)}{\cap}} \to \phi_2(\mathcal{U}_1 \cap \mathcal{U}_2) \text{ holomorphic}$$

(which implies $\phi_1 \circ \phi_2^{-1}$ holomorphic)

resulting composition is called "transition function"

Check in non-trivial example of S^2: $\mathcal{U}_1 \cap \mathcal{U}_2 = S^2 \setminus \{(0,0,\pm 1)\}$

$$\phi_2 \circ \phi_1^{-1} = \phi_2 \left(\frac{2\,\mathrm{Re}(z)}{|z|^2+1}, \frac{2\,\mathrm{Im}(z)}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1} \right) = \frac{\mathrm{Re}(z)}{|z|^2} - i\,\frac{\mathrm{Im}(z)}{|z|^2}$$

$\phi_1(S^2 \setminus \{(0,0,\pm 1)\}) = \mathbb{C} \setminus \{0\}$, so $1/z$ holom. on this set. $= \dfrac{\bar{z}}{|z|^2} = \dfrac{1}{z}$

Such a compatible system of charts $\{ \phi_\alpha : U_\alpha \to V_\alpha \}$ s.t. $\bigcup_\alpha U_\alpha = X$
(pairwise)

is called a complex atlas. (Here we say that if $U_\alpha \cap U_\beta = \phi$ then their charts are automatically considered compatible, so can ask this of any two charts.)

Say that two atlases are equivalent if each chart $\phi : U \to V$ of one is compatible with every chart of the other.

An equivalence class of complex atlases is called a complex structure.

Finally we define a Riemann surface as a (second-countable, connected, Hausdorff) topological space X with a complex structure.

(Hausdorff : topology fine enough to separate points : i.e. contained in separate nbhds

second-countable : has basis of topology that is countable)

[usually just stipulate that system of charts is countable. Miranda notes countable charts $\Rightarrow$ X is second-countable]

Abbreviated definition : A Riemann surface is a (connected) 1-dim'l cx manifold.

The adjective "complex" is key - any time we impose additional structure on objects, want maps to have it as well.

so when we define

topological manifold : transition maps are continuous

C^∞ - manifold : —"— are smooth (∞-ly differentiable)

cx manifold : transition maps are holomorphic.

Since real, imag parts of holomorphic function are infinitely differentiable then Riemann surfaces are also examples of 2-dimensional real C^∞-manifolds, simply rewriting cx. charts $\phi : \mathcal{U} \to V \subset \mathbb{C}$ as $(\mathrm{Re}(\phi), \mathrm{Im}(\phi)) : \mathcal{U} \to V \subset \mathbb{R}^2$.

Note also that transition functions $T = \phi_2 \circ \phi_1^{-1}$ never have 0 derivative on their domain of definition. – usual pf: T invertible, then $T(S(w)) = w$ with inverse S

take derivs:

$$T'(S(w))\, S'(w) = 1$$

so transition functions are always conformal.

This means we can define an orientation on X as follows: define one on $\mathcal{U}_\alpha$ by pulling back orientation on $V_\alpha \subset \mathbb{C}$ by ϕ_α^{-1}

It is well-defined because all charts $\phi_{\alpha'}$ differ by conformal map which is angle preserving. Compatibility of charts on intersections $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$ allow us to extend to a global orientation on X.

Finally any topological space is called <u>compact</u> if it satisfies the Heine-Borel property – every open cover admits a finite subcover.

<u>Examples</u>: Riemann sphere, $\mathbb{C}/M$ are compact Riemann surfaces. (not using Heine-Borel to check this. Rather equivalent notions for these particular topologies in $\mathbb{R}^3$ and in $\mathbb{C}$.)

From our discussion above, compact Riemann surfaces are compact, orientable, C^∞-real manifolds of dim 2.

<u>Thm</u>: Every such 2-manifold is diffeomorphic to a g-holed torus for some $g \geq 0$.

$g=0$: sphere

$g=1$: torus

if $g \geq 2$ obtained by attaching additional "handles" to the surface