Suppose $\mathrm{Im}(\tau') \geq \mathrm{Im}(\tau) \Rightarrow |c\tau + d| \leq 1$, which is very restrictive if $c, d \in \mathbb{Z}$.

easy case method to show τ unique.

Another consequence of case method: Very few pairs (ω_1, ω_2) achieve the desired τ. Basically 2: (ω_1, ω_2) minimal, and $(-\omega_1, -\omega_2)$

except for two special cases: $\tau = i$, $\tau = e^{2\pi i/3}$ (fixed points of unimodular transf.)

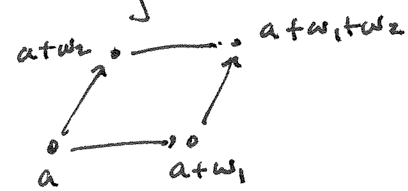
$\tau \mapsto -1/\tau$ $\qquad$ $\tau \mapsto -(\tau+1)/\tau$

$\tau \mapsto -1/\tau+1$

Do doubly periodic functions exist? doubly-periodic = elliptic

No non-trivial doubly-periodic analytic functions.

pf: A doubly periodic function f is defined by its values on any parallelogram with vertices $\{a, a+\omega_1, a+\omega_2, a+\omega_1+\omega_2\}$



closure of this region is compact, so f bounded on //-ogram, and hence on entire plane.

By Liouville's thm, any such function is constant.

What about meromorphic functions? Only finitely many poles in any fundamental parallelogram, since poles isolated.

Proposition: Sum of residues of elliptic function in fundamental //-ogram is 0.

pf: Pick parallelogram P so that the boundary ∂P contains no poles.

Calculate $\displaystyle \frac{1}{2\pi i}\int_{\partial P} f(z)\,dz =$ Clear by periodicity that opposite sides of ∂P cancel so integral $= 0$. //

Corollary: # of poles = # of zeros for elliptic function in fundamental //-ogram.

pf: If f is elliptic, so is f'/f. Apply previous result to f'/f, remembering this function records zeros and poles (with opposing signs)

Proposition (stronger version of previous corollary)

zeros (counted according to mult.) : $\{a_i\}_{i=1}^n$ (n same by previous corollary)

poles (counted according to mult.) : $\{b_i\}_{i=1}^n$

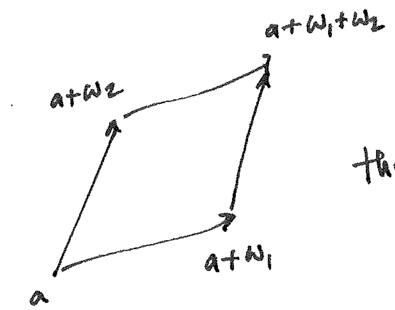
then $a_1 + \cdots + a_n \equiv b_1 + \cdots + b_n$ (i.e. differ by an elt. of M)

pf: Again choose fundamental //-ogram P which avoids zeros, poles.

Consider contour integral $\displaystyle \frac{1}{2\pi i}\int_{\partial P} \frac{z f'(z)}{f(z)} = a_1 + \cdots + a_n - b_1 - \cdots - b_n$

Cauchy's thm.

If P:



then $$\int_{\partial P} = \underbrace{\int_a^{a+\omega_1}}_{1} - \underbrace{\int_{a+\omega_2}^{a+\omega_1+\omega_2}}_{2} \left(z\frac{f'(z)}{f(z)}\right)dz$$

$$+ \underbrace{\int_{a+\omega_1}^{a+\omega_1+\omega_2}}_{3} - \underbrace{\int_a^{a+\omega_2}}_{4} \left(z\frac{f'(z)}{f(z)}\right)dz$$

make change of vars in 2:

$z \mapsto z - \omega_2$.

then using periodicity of f, f' with respect to ω_2,

this becomes $\displaystyle \frac{\omega_2}{2\pi i} \int_a^{a+\omega_1} \frac{f'(z)}{f(z)}\, dz$ ~ $\omega_2 \cdot$ winding # of image of line between $a, a+\omega_1$ under f.

Similarly, the other pair gives $\omega_1 \cdot$ (integer) $= \omega_2 \cdot$ (integer).

so result is in M. //

In particular, we've shown that an elliptic function can't have a single simple pole in the fundamental parallelogram, since sum of residues is 0.

Next simplest scenario: one double pole at single point in parallelogram
i.e. pole of order 2

Naive guess: $\displaystyle \sum_{\omega \in M} \frac{1}{(z-\omega)^2}$, but this won't converge when we sum over a lattice.

We saw last semester that we can dictate singular parts of poles in merom. function but may need to subtract terms from Taylor expansion of singular part at origin. (except for $\frac{1}{z^2}$, which we extract separately) Remove constant term.

Analyze absolute convergence:

$$\left| \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right| = \left| \frac{z(2\omega - z)}{\omega^2 (z-\omega)^2} \right| .$$

Bound this for $|\omega|$ sufficiently large.

For z: fixed, we choose $|\omega| > 2|z|$

Write $\left|\frac{z(2w-z)}{w^2(z-w)^2}\right| \leq \left|\frac{zw}{w^2(z-w)^2}\right| + \left|\frac{z(w-z)}{w^2(z-w)^2}\right|$ $(*)$

Since $|w| > 2|z|$ then $|z-w| > \frac{1}{2}|w|$ (since $|z-w| \geq |w| - |z| > 1/2|w|$)

So $(*) < \left|\frac{zw}{w^2(\frac{1}{2}w)^2}\right| + \left|\frac{z}{w^2(\frac{1}{2}w)}\right| = \frac{|z|}{|w^3|} \cdot 6$

Thus to show uniform convergence on any compact set, want to check

$$\sum_{w\in M, w\neq 0} \frac{1}{|w|^3} \text{ converges}$$ (result then follows by Weierstrass M-test.)

to show this: Given $n \in \mathbb{N}$, consider parallelogram

$$P(n) = \{ a_1 w_1 + a_2 w_2 \mid a_1, a_2 \in \mathbb{R}, \max(|a_1|, |a_2|) = n \}$$

The boundary $\partial P(n)$ contains $8n$ points. Minimal abs. value of these is $\geq kn$ where k: shortest distance from 0 to $\partial P(1) \cap M$.

So points on $\partial P(n)$ contribute less than: $8n / k^3 n^3$ to sum.

Summing over n, this converges. //

Still need to show periodicity of $\wp(z) = \frac{1}{z^2} + \sum_{\substack{w\in M\\ w\neq 0}} \frac{1}{(z-w)^2} - \frac{1}{w^2}$:

Take derivative $\wp'(z) = -2 \sum_{w\in M} \frac{1}{(z-w)^3}$. So $z \mapsto z+w$ just permutes terms in sum (ok to rearrange)

hence $\wp'(z)$ elliptic. (Note can take derivs term by term as uniform conv. on compacta)