periodic functions: period is cx. number ω s.t. $f(z+\omega) = f(z)$ $\forall z \in \Omega$ ①

(want $z \in \Omega \Rightarrow z \pm \omega \in \Omega$)

Note if ω is a period, then so is $n\omega$, $n \in \mathbb{N}$.

e.g. 2ω: $f(z+\omega+\omega) = f(z+\omega) = f(z)$

periodicity wrt ω used in both places

Similarly, if ω_1 is period, ω_2 is period then so is $\omega_1 + \omega_2$.

so set of periods M has structure of $\mathbb{Z}$-module, for any f.

M can't have an accumulation point, else f is just constant. (assuming f meromorphic on $\mathbb{C}$)

so M consists of isolated points (we say such sets are "discrete")

Thm: Given any f meromorphic on $\mathbb{C}$, its period module is free $\mathbb{Z}$-module.

i.e. periods are 0, or $\{n\omega\}_{n \in \mathbb{N}}$ for some $\omega \in \mathbb{C}$, or $\{n_1\omega_1 + n_2\omega_2\}_{n_1, n_2 \in \mathbb{N}}$ for some $\omega_1, \omega_2 \in \mathbb{C}$

pf: If $\neq 0$, pick $\omega \in M$ minimal w.r.t. abs. value.
(might be multiple points with same abs. value, but only finitely many since $|z| = r$ is compact)

Any other period ω', not an integral multiple of ω, can't lie on same line as ω through origin (i.e. $\frac{\omega}{\omega'}$ not real) If it were real we could find period smaller than ω: say $n < \frac{\omega'}{\omega} < n+1$ for some integer n,

then $|n\omega - \omega'| < |\omega|$ ↯.

Only remaining possibility ω' not on line through origin containing ω.

Rename ω, ω' as ω_1, ω_2. Want to show all periods must be of the form $n_1\omega_1 + n_2\omega_2$.

ω_1, ω_2 is $\mathbb{R}$-basis for $\mathbb{C}$: any cx. # is $\lambda_1\omega_1 + \lambda_2\omega_2$, $\lambda_1, \lambda_2 \in \mathbb{R}$.

(geometrically clear. Algebraically, given $\omega \in \mathbb{C}$, then want sol'ns to

$$\omega = \lambda_1\omega_1 + \lambda_2\omega_2$$
$$\bar\omega = \lambda_1\bar\omega_1 + \lambda_2\bar\omega_2$$

to guarantee λ_1, λ_2 real.

indeed has unique sol'n since $\omega_1\bar\omega_2 - \bar\omega_1\omega_2 \neq 0$ else ω_1/ω_2 real)

Pick ω' minimal such M not on the line $\mathbb{R}\omega$.

Given any $\omega_3 \in M$, again show $\omega_3 = n_1\omega_1 + n_2\omega_2$ by contradiction.

If not, say $\omega_3 = \lambda_1\omega_1 + \lambda_2\omega_2$ λ_1, λ_2 not both real, pick m_1, m_2 s.t. $|\lambda_1 - m_1| \le 1/2$, $|\lambda_2 - m_2| \le 1/2$

then $\omega_3 - m_1\omega_1 - m_2\omega_2 =: \omega_4$ has $|\omega_4| < \frac{1}{2}|\omega_1| + \frac{1}{2}|\omega_2| \le |\omega_2|$

strict b/c otherwise $\frac{\omega_1}{\omega_2}$ real.

which gives another contradiction.

Proposition: Any two bases $\{\omega_1, \omega_2\}$, $\{\omega_1', \omega_2'\}$ for M are related by

elt. of $GL(2,\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \begin{array}{c} ad - bc = \pm 1 \\ a,b,c,d \in \mathbb{Z} \end{array} \right\}$ by $\begin{pmatrix} \omega_2' \\ \omega_1' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \omega_2 \\ \omega_1 \end{pmatrix}$

invertible matrices with entries in $\mathbb{Z}$

pf: clear that $\exists$ $a,b,c,d \in \mathbb{Z}$ s.t. $\omega_2' = a\omega_2 + b\omega_1$, $\omega_1' = c\omega_2 + d\omega_1$ by ~~[illegible]~~ def'n of $\mathbb{Z}$-bases

But similarly $\exists$ a', b', c', d' s.t. $\begin{pmatrix} \omega_2 \\ \omega_1 \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\begin{pmatrix} \omega_2' \\ \omega_1' \end{pmatrix}$ so composing,

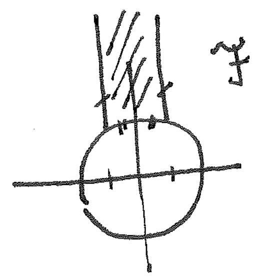
$$\begin{pmatrix}\omega_2\\ \omega_1\end{pmatrix} = \begin{pmatrix}a' & b'\\ c' & d'\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\omega_2\\ \omega_1\end{pmatrix}$$

Can we conclude that product is identity? Yes.

Easiest to see by noting same relation holds for $\begin{pmatrix}\bar{\omega}_2\\ \bar{\omega}_1\end{pmatrix}$, and again $\omega_2\bar{\omega}_1 - \omega_1\bar{\omega}_2 \neq 0$ if ω_1/ω_2 not real, so $\begin{pmatrix}\omega_2 & \bar{\omega}_2\\ \omega_1 & \bar{\omega}_1\end{pmatrix}$ invertible and can "cancel" it from both sides //

Thm: Given period module M, $\exists$ $\{\omega_1, \omega_2\}$ basis s.t.

$\frac{\omega_1}{\omega_2} = \tau$ is a point in:



i.e. (a) $\mathrm{Im}(\tau) > 0$

(b) $-\frac{1}{2} < \mathrm{Re}(\tau) \leq \frac{1}{2}$

(c) $|\tau| \geq 1$

(d) $\mathrm{Re}(\tau) \geq 0$ if $|\tau| = 1$

pf sketch: pick ω_1, ω_2 minimal as before. Then $\tau = \omega_2/\omega_1 \in \bar{\mathcal{F}}$.

$\left(|\omega_1| \leq |\omega_2|,\ |\omega_2| \leq \min\{|\omega_1+\omega_2|, |\omega_1-\omega_2|\} \right)$

$\Rightarrow$ (c), (b)

If $\mathrm{Im}(\tau) < 0$, then just replace (ω_1, ω_2) by $(-\omega_1, \omega_2)$.

If $\mathrm{Re}(\tau) = -\frac{1}{2}$ replace (ω_1, ω_2) by $(\omega_1, \omega_1+\omega_2)$

If $|\tau| = 1$, $\mathrm{Re}(\tau) < 0$ replace (ω_1, ω_2) by $(-\omega_2, \omega_1)$.

Could there be two values $\tau, \tau' \in \mathcal{F}$ for two choices of bases of M?

If so, then $\tau' = \dfrac{a\tau+b}{c\tau+d} \Rightarrow \mathrm{Im}(\tau') = \mathrm{Im}(\tau) / |c\tau+d|^2$

Suppose $\mathrm{Im}(\tau') \geq \mathrm{Im}(\tau) \Rightarrow |c\tau + d| \leq 1$, which is very restrictive if $c, d \in \mathbb{Z}$.

easy case method to show τ unique.

Another consequence of case method: Very few pairs (ω_1, ω_2) achieve the desired τ. Basically 2: (ω_1, ω_2) minimal, and $(-\omega_1, -\omega_2)$ except for two special cases: $\tau = i$, $\tau = e^{2\pi i/3}$ (fixed points of unimodular transf.)

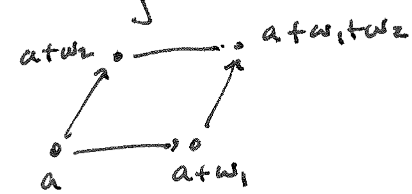
$\tau \mapsto -1/\tau$ $\tau \mapsto -(\tau+1)/\tau$

$\tau \mapsto -1/\tau+1$

Do doubly periodic functions exist? doubly-periodic = elliptic

No non-trivial doubly-periodic analytic functions.

pf: A doubly periodic function f is defined by its values on any parallelogram with vertices $\{a, a+\omega_1, a+\omega_2, a+\omega_1+\omega_2\}$



closure of this region is compact, so f bounded on //-ogram, and hence on entire plane.

By Liouville's thm, any such function is constant.

What about meromorphic functions? Only finitely many poles in any fundamental parallelogram, since poles isolated.

Proposition: Sum of residues of elliptic function in fundamental //-ogram is 0.

pf: Pick parallelogram P so that the boundary ∂P contains no poles.