Remember $\xi \mapsto z = \underbrace{\dfrac{R(R\xi + a)}{R + \bar{a}\xi}}_{L(\xi)}$ maps unit disk $|\xi| \le 1$ to $|z| \le R$ with $0 \mapsto a$.

Then if $u$ is harmonic on closed disk $|z| \le R$, then

$u(L(\xi))$ is harmonic for $|\xi| \le 1$ and

$$u(a) = \frac{1}{2\pi} \int_{|\xi|=1} u(L(\xi))\, d(\arg \xi) \qquad (*)$$

More explicitly, we invert $L$, $\xi = \dfrac{R\cdot(z-a)}{R^2 - \bar{a}z}$. Then $d(\arg \xi)$,

which is $-i\dfrac{d\xi}{\xi} = -i\left(\dfrac{1}{z-a} + \dfrac{\bar{a}}{R^2 - \bar{a}z}\right) dz = \left(\dfrac{z}{z-a} + \dfrac{\bar{a}z}{R^2 - \bar{a}z}\right) d\theta$

[indeed $\xi = e^{i\phi}$, then $d\xi = i e^{i\phi} d\phi$ i.e. $d\phi = -i\dfrac{d\xi}{\xi}$ ]

Now $|z| = R$ in $(*)$, so $\dfrac{\bar{a}z}{R^2 - \bar{a}z} = \dfrac{\bar{a}}{\bar{z} - \bar{a}}$ and $\dfrac{z}{z-a} + \dfrac{\bar{a}}{\bar{z}-\bar{a}} = \dfrac{R^2 - |a|^2}{|z-a|^2}$
$\Rightarrow R^2 = z\bar{z}$

$$= \operatorname{Re}\left(\frac{z+a}{z-a}\right)$$

Thus we may rewrite $(*)$ in the form:

$$u(a) = \frac{1}{2\pi} \int_{|z|=R} \frac{R^2 - |a|^2}{|z-a|^2}\, u(z)\, d\theta \qquad \text{"Poisson's formula"}$$

or can rewrite using polar coordinates as in Ahlfors. p. 167.

Remarks: Poisson's formula is still true under weaker condition that $u(z)$ is harmonic on open disk, and only continuous on closed disk. Reason: for $0 < r < 1$ real, then $u(rz)$ is harmonic on closed unit disk with

$$u(ra) = \frac{1}{2\pi} \int_{|z|=R} \frac{R^2 - |a|^2}{|z-a|^2} u(rz)\, d\theta .$$

Take limit as $r \to 1$. $u(z)$ uniformly continuous on compact $|z| \leq R$ so $u(rz) \to u(z)$ uniformly as $r \to 1$ for $|z| = R$.

- Use Poisson's formula to give formula for harmonic conjugate.

Write $u(z) = \operatorname{Re}\left[ \underbrace{\frac{1}{2\pi i} \int_{|\xi|=R} \frac{\xi+z}{\xi-z} u(\xi) \frac{d\xi}{\xi}}_{} \right]$ (**)

defines analytic function for $|z| < R$ if $u(\xi)$ is piecewise continuous. on $|\xi| = R$

- Also shows that construction ensures $u(z)$ harmonic in disk.

(We assumed $u$ harmonic on disk. In Dirichlet problem, don't necessarily know solution exists)

Thm 23 p.169 in Ahlfors.

Remaining annoyance: if we define $u$ by (**), does it have the correct boundary values at $|z| = R$

Given piecewise continuous function $u(\theta)$, $0 \le \theta \le 2\pi$

define $$P_u(z) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\, \frac{e^{i\theta}+z}{e^{i\theta}-z}\, u(\theta)\, d\theta$$

"Poisson integral of $u$" ( Poisson's formula with $R=1$ so points given by $e^{i\theta}$ )

Thm: $$\lim_{z \to e^{i\theta_0}} P_u(z) = u(\theta_0)$$ provided $u$ is continuous at $\theta_0$.

pf. As a function of $u$, $P_u$ is linear functional:

① $P_{u_1+u_2} = P_{u_1} + P_{u_2}$

② $P_{c\cdot u} = c \cdot P_u$.

$\Big\}$ def'n of linear fn.

AND ③ $P_c = c$ $\quad c$: constant function.

( since $u(z) = c$ is harmonic, so $u(0) = c = P_u$ ) using Poisson's formula

④ $u \geq 0 \Rightarrow P_u(z) \geq 0$.

since $$\mathrm{Re}\, \frac{e^{i\theta}+z}{e^{i\theta}-z} = \frac{1-|z|^2}{|e^{i\theta}-z|^2}$$

so positive for $|z|<1$.

$\Rightarrow$ if $m \le u \le M$ then

$m \le P_u \le M$. (*)

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$C_1, C_2$ be "complementary arcs" on unit circle

$C_1 \cup C_2 = C$

Given $u$, write

$$u_1(\theta) = \begin{cases} u & \text{if } \theta \in C_1 \\ 0 & \text{else} \end{cases}$$

$$u_2(\theta) = \begin{cases} u & \text{if } \theta \in C_2 \\ 0 & \text{else} \end{cases}$$

By linearity, $P_u = P_{u_1} + P_{u_2}$.

Let $C_1$: closed arc

$C_2$: open arc.

Writing $P_{u_1}$ as line integral over $C_1$, defines a harmonic function.
(Same reasoning as before: Real part of analytic function)

Moreover $\frac{1-|z|^2}{|e^{i\theta}-z|^2}$ vanishes for all $z \neq e^{i\theta}$, so $P_{u_1}(z) = 0$ $\forall z \in C_2$, as integral domain is $C_1$

and $P_{u_1}(z) \to 0$ as $z \to e^{i\theta_0}$ in $C_2$

Given any $\theta_0$, assume $u(\theta_0) = 0$
(otherwise replace $u$ by $u - u(\theta_0)$)

Pick $C_2$ so that $e^{i\theta_0}$ is an interior point, and that $\theta \in C_2$ are close enough to $\theta_0$ so that $|u(\theta)| < \epsilon/2$ if $e^{i\theta} \in C_2$.
(possible since $u$ continuous at $\theta_0$)

Then $|u_2(\theta)| < \epsilon/2$ since $u = u_2$ on $C_2$

$\Rightarrow$ $|P_{u_2}(z)| < \epsilon/2$ $\forall$ $|z|<1$ by $(*)$

From above analysis $P_{u_1}(z) \to 0$ as $z \to e^{i\theta_0}$ in $C_2$
So pick $\delta$ s.t. $|P_{u_1}(z)| < \epsilon/2$ for $|z - e^{i\theta_0}| < \delta$.

$\Rightarrow$ $|P_u(z) - 0| < \epsilon$ when $|z|<1$ and $|z - e^{i\theta_0}| < \delta$. //