Goals for course:

Material — Conformal maps / harmonic functions (Ahlfors) 3.2 / Ch. 6
3.3

Riemann surfaces (Miranda)

(Do we do digression on elliptic functions in between?)

Advance mathematical timeline: Move from 19^{th} / early 20^{th} century results to mid 20^{th} century results.

Last semester: Showed analytic functions / merom. functions behaved like polynomials / rational functions.

(isolated zeros, derivs. of all orders, representable by power series, limited growth possibilities)

All Consequences of Cauchy Integral formula, so natural to continue with applications of integration.

- other applications of analytic functions → analytic functions respect geometry of space.

- source for producing analytic functions.

→ rep'n of functions, analytic continuation

→ Riemann surfaces.

Definition: Ω : region, $f : \Omega \to \mathbb{C}$ is conformal at $z_0 \in \Omega$

if $\exists\ \theta \in [0, 2\pi)$, $r > 0$ s.t., for any (differentiable ~~smooth~~) curve $\gamma := \gamma(t)$ in Ω

with $\gamma(t_0) = z_0$, $\gamma'(t_0) \neq 0$

then
$$\underbrace{\left| \frac{d}{dt} \left(f(\gamma(t)) \right) \right|_{t=t_0}}_{} = r \cdot |\gamma'(t_0)|$$

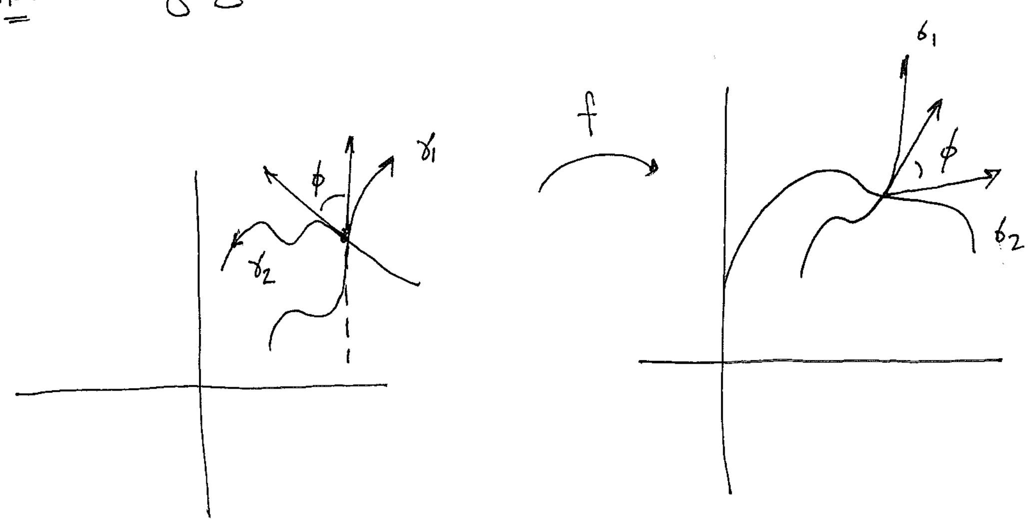
define $\sigma(t) = f(\gamma(t))$
write $\sigma'(t_0)$ instead.

and $\arg \sigma'(t_0) \equiv \arg \gamma'(t_0) + \theta \pmod{2\pi}$

f is conformal on Ω if conformal at every pt. $z_0 \in \Omega$.

i.e. f acts merely by rotating and stretching tangent vectors, but not changing their angles, + stretch factor is same in all directions

Picture:



pf: immediate from chain rule.

Proposition: $f : \Omega \to \mathbb{C}$ analytic with $f'(z) \neq 0$, then f is conformal at z_0 with $\theta = \arg(f'(z_0))$, $r = |f'(z_0)|$

i.e. $\quad \sigma'(t_0) = f'(z_0) \cdot \gamma'(t_0)$ //

take modulus, arg.

Converse also true: if f satisfies

$$\left| \frac{d}{dt} \left(f(\gamma(t)) \right|_{t=t_0} \right| = r \cdot |\gamma'(t_0)| \quad \text{for all curves } \gamma \text{ through } z_0.$$

and $\quad \arg \left(\frac{d}{dt} \left(f(\gamma(t)) \right|_{t=t_0} \right) \equiv \arg \gamma'(t_0) + \theta$

then $f'(z_0)$ exists and is equal to $re^{i\theta}$.

Or even just angle preserving + continuous first partials $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$.

then $\quad \sigma'(t_0) = \left.\frac{\partial f}{\partial x}\right|_{z=z_0} x'(t_0) + \left.\frac{\partial f}{\partial y}\right|_{z=z_0} y'(t_0)$

$$= \frac{1}{2}\left(\frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right) \gamma'(t_0) + \frac{1}{2}\left(\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right) \overline{\gamma'(t_0)}$$

If $\arg \left(\frac{\sigma'(t_0)}{\gamma'(t_0)} \right)$ constant, ~~...~~ indep. of $\gamma'(t_0)$, then

$$\Rightarrow \frac{1}{2}\left(\frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right) + \frac{1}{2}\left(\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right) \frac{\overline{\gamma'(t_0)}}{\gamma'(t_0)} \qquad (*)$$

has const argument. Varying $\arg(\gamma'(t_0))$, $(*)$ traces out points on circle with radius $\frac{1}{2}\left(\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right)$. Hence $(*)$ is const. indep. of $\arg(\gamma'(t_0))$ only if $\frac{\partial f}{\partial x} = -i \frac{\partial f}{\partial y}$ i.e. C.R. equns. //

Why is this useful? Harmonic functions: twice-differentiable real-valued functions $u : \Omega \to \mathbb{R}$ s.t. (Ω: open in plane)

$$\nabla^2 u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Recall that if $f = u + iv$ is analytic, then u, v harmonic since just differentiate C-R eqns and use equality of mixed partials.

(note f analytic $\Rightarrow$ u, v infinitely diff.)
so second partials exist, continuous.

Physical significance: u harmonic then $u(x,y) = c$ gives flow lines for solutions to natural problems:

- flow of irrotational, inviscid, incompressible plane flow
- flow in planar electrostatic / magnetostatic field (force lines)
- (steady-state) heat flow in uniform plate (modeled in plane)

Proposition: u, v harmonic conjugates on Ω. Suppose $u(x,y) = c_1$, $v(x,y) = c_2$ define smooth curves. Then these curves intersect orthogonally.

pf : $u(x,y) = c_1$ smooth if $\text{grad}(u) = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) \neq 0$ for (x,y) with $u(x,y) = c_1$

(and grad(u) is perpendicular to tangent vector at u.)

just chain rule applied to

$$\underbrace{\frac{d}{dt}\left(u(x(t), y(t)\right)}_{\text{grad}(u)\cdot (x'(t), y'(t))} = \frac{d}{dt}(c_1) = 0$$

show that normal vectors of u, v perpendicular.

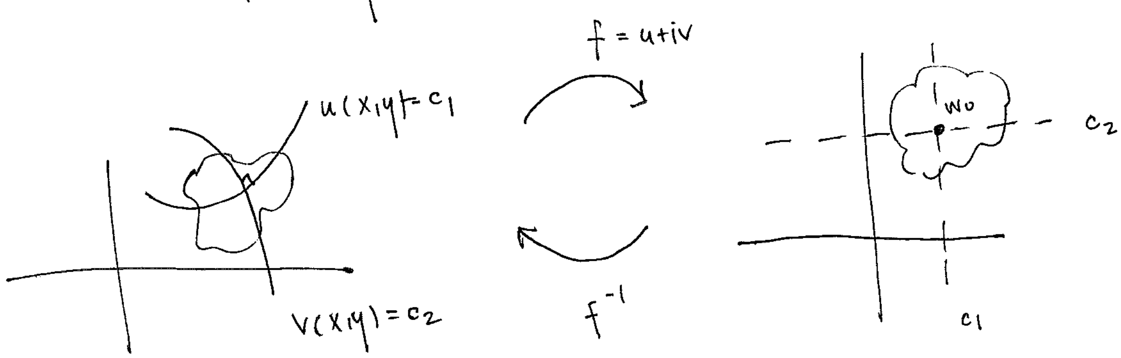
i.e. compute $\text{grad}(u) \cdot \text{grad}(v) = \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial y} = 0$

by C-R eqns.

pf 2 : f analytic, $f'(z_0) \neq 0$, then

f^{-1} analytic in nbhd. of $w_0 = f(z_0)$ with $f^{-1\prime}(w_0) \neq 0$ by inverse function thm., hence conformal at w_0.

cx inv. function thm ~~...~~ real-var follows from inv. fun. thm + Cauchy-Riemann eqns.



But $u(x,y) = c_1$ is image of line c_1 under f^{-1}

Consequence: interesting flows arise from harmonic functions, and hence from analytic functions f by taking re, im parts.

Moreover, harmonic conjugates give flow and corresponding level curves

"equipotential lines"
"isothermal lines"
in the language of these examples.

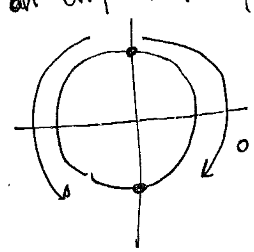
conformal maps enrich our collection of interesting flows.

Allow us to transfer problems from difficult flows to simpler flows.

e.g. Dirichlet problem. Contrived example:

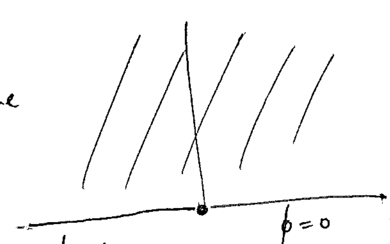
Find harmonic function ϕ on unit disk $|z|<1$

with values on unit circle for ϕ:



Use conformal map

$$W = -i\,\frac{z+i}{z-i}$$ maps circle to upper half plane

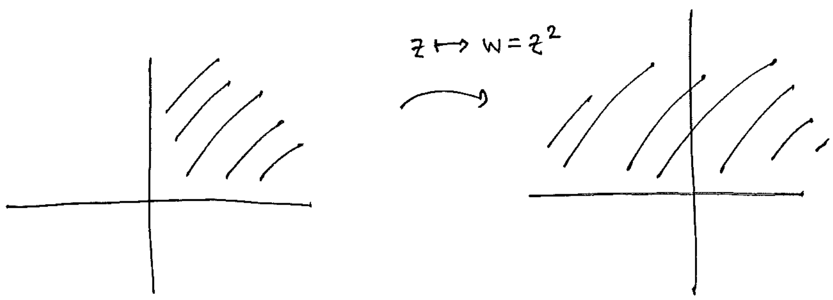


$$\phi = \frac{1}{\pi}\arg\left(\frac{z+i}{z-i}(-i)\right)$$

so have solution $\phi^* = \frac{1}{\pi}\arg(w)$

Ex

Example : $f(z) = z^2$. $f'(z) = 2z$, and indeed f fails to be conformal at $z = 0$.



$f = u + iv$ with $u = x^2 - y^2$, $v = 2xy$ are harmonic functions.

(check explicitly that derivative condition satisfied)

u, v s.t. $f = u + iv$ analytic are referred to as "harmonic conjugates"