Consider inverse functions —

Define $z = \log w$ (here thinking of z as dependent var.)

where $e^z = w$

Problems : 1 We noted before $e^z \neq 0 \ \forall z \in \mathbb{C}$, so $\log(0)$ not defined

2 if $w \neq 0$, then using $|w| = |e^z| = e^x$,

we have $e^{iy} = e^z / e^x = w/|w|$

In equation $\underset{(\text{in } x)}{|w| = e^x} \Rightarrow x = \log|w|$ (unique solution, according to real logarithm.)

Now equation $e^{iy} = w/|w|$ (in y)

has a solution in every interval of size 2π, since $w/|w|$ is indeed on unit circle. i.e. ∞-ly many solutions $y \in \mathbb{R}$.

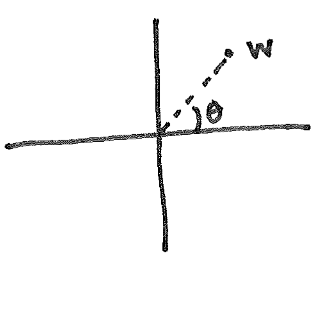
Conclusion. For $w \neq 0$, ∞-ly many possibilities for $\log(w)$.

"multi-valued function"

write $\log w = z = x + iy$ then $x = \log|w|$ (real log)

$y := \arg(w)$

angle measured counter-clock. from real axis



then $\log w = x + iy$

with $y = \theta$ or $\theta + 2\pi k$
$k \in \mathbb{Z}$

(e.g. $w = 1+i$, $\log(w) \equiv \log\sqrt{2} + i\frac{\pi}{4} \pmod{2\pi i}$)

Consequence : Care must be taken in manipulation of multi-valued functions

Which of the following identities is true? (for $z \neq 0$)

$$e^{\log z} = z \quad \text{or} \quad \log(e^z) = z$$

this is ok since $e^{2\pi i} = 1$
so LHS same for any
choice of $\arg(z)$.

this is only an identity
mod $2\pi i$
(i.e. true up to integer multiple of $2\pi i$)

Similarly, given $e^{\log(\alpha\beta)} = \alpha\beta = e^{\log(\alpha)} e^{\log(\beta)} = e^{\log(\alpha)+\log(\beta)}$

$$\Rightarrow \quad \log(\alpha\beta) \equiv \log(\alpha) + \log(\beta) \pmod{2\pi i}$$

or we could use the (somewhat dangerous) notation: (as in Ahlfors)

$\log(\alpha\beta) = \log(\alpha) + \log(\beta)$ to mean that
the two sides agree as sets since both represent same collection of
$2\pi i$ multiples of some cx. number.

Play similar game to show $\arg(\alpha\beta) \equiv \arg(\alpha) + \arg(\beta)$. $(2\pi i)$

To consider exponentials / logs assoc. to bases other than e:

$a, b \in \mathbb{C}$, $a \neq 0$: form $a^b := \exp(b \log a)$

e.g. $2^i = e^{i \log 2}$ but $\log 2 = \underbrace{\log|2|} + i \overbrace{\arg(2)}^{0 + 2\pi k}$ so...

Some authors write $\operatorname{Log} 2$ with Log denoting real logarithm.

$$\ldots = e^{i \operatorname{Log} 2} \cdot e^{-2\pi k}$$

in particular, using various choices of k,
can make 2^i arbitrarily
large or small in magnitude.

Ahlfors' work-around: if base a is real, positive,

then take $\log(a)$ to mean real logarithm. (i.e. take $k=0$)

But can't avoid this issue for a neg or a (genuinely) complex base a.

Even simple equations can be false: $\sqrt{z}$ is a 2-valued function for any $z \neq 0$.

So $\sqrt{1} + \sqrt{1} = 2 \cdot \sqrt{1}$ is false since ± 1 (i.e. 1 and $1 \cdot e^{2\pi i/2}$)

$\Rightarrow$ LHS takes values $0, \pm 2$, RHS only takes values ± 2.

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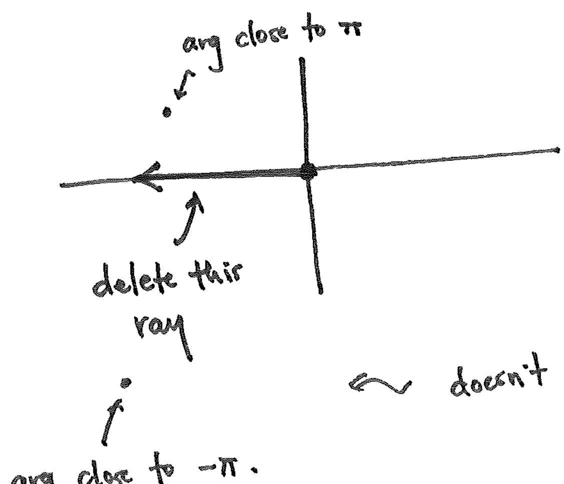
inverse trig functions look more complicated

e.g. $\arccos w = -i \log\left(w \pm \sqrt{w^2-1}\right)$

Fix: Consider restricted region (open set) of plane, and choose values at each point such that multi-valued function becomes continuous single-valued function. "branch"

example: For logarithm, largest possible branch delete a ray from origin:



force $\arg(z) \in (-\pi, \pi)$

doesn't violate continuity, since ray removed.

Example 2 : $w = \sqrt{z}$ by $w^2 = z$ is doubly-valued function.
(for all $z \neq 0$)

Two solutions differ by $(-1) = e^{\pi i}$: rotation by $180°$ about origin.

Can we just choose w with positive real part ? Problem for purely imag. roots. So omit neg. real axis (including 0).

Is this continuous ? Check $\lim_{z_1 \to z_2} \overbrace{w(z_1)}^{w_1} = \overbrace{w(z_2)}^{w_2}$ (i.e. $|w_2 - w_1| \to 0$ as $|z_2 - z_1| \to 0$)

$w_1 = u_1 + iv_1$, $w_2 = u_2 + iv_2$. then
$$|z_1 - z_2| = |w_1^2 - w_2^2|$$
$$= |w_1 - w_2||w_1 + w_2|$$

and $|w_1 + w_2| \geq u_1 + u_2 > u_1$ since $u_i > 0$.

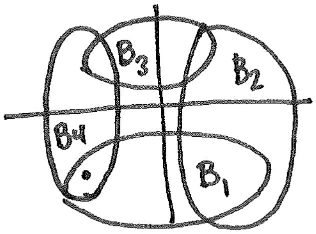
So $|w_1 - w_2| < \dfrac{|z_1 - z_2|}{u_1}$ u_1: fixed in limit as $z_1 \to z_2$ so indeed
$$\lim_{z_1 \to z_2} |w_1 - w_2| = 0.$$

What is derivative? Implicit diff.

$$z = w^2 \rightarrow \frac{dw}{dz} = \frac{1}{2w}.$$

see p. 70 for similar treatment of $\log z$'s branch in Example 1.
Ahlfors

Riemann surface: In our example of $\log z$:



Branches B_i will be uniquely determined if they are to agree with function on B_{i-1} (and be continuous)

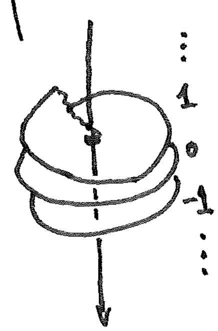
But in our picture, if $z \in B_1 \cap B_4$ then

$\log z$ for $B_1 \neq \log z$ for B_4 (differ by $2\pi i$)

Can distinguish them by pair (z, n) where n records multiple of $2\pi i$.

"Riemann surface for $\log z$" (not going to give formal definition at moment)
(obtained by taking successive principal branches, by removing real or imag. half axes)

Kind of suggestive picture:



sheets of $\mathbb{C}$ stacked above each other, labeled by their corresp. $n \in \mathbb{Z}$.

(In general may be hard or impossible to draw Riemann surfaces in space)

Thm: function $\log z$ is analytic at every pt. in Riemann surface and satisfies $\frac{d}{dz} \log z = \frac{1}{z}$

e.g. $z^{1/3}$. ← triply valued away from 0.

$z \longmapsto \log z =: w$ is 1-1 mapping from Riemann surface to w-plane $\mathbb{C}$, and if we fix $\log 1 = 0$, then inverse is e^z.

Comments about pf:

Analyticity just follows from taking nbhd. of any pt. contained in a branch.

For inverses: we knew that $e^{\log z} = z$ already.

Issue was with $\log(e^z)$. Let $\phi(z) = \log(e^z)$.

$\phi'(z) = 1$ by chain rule, so $\phi(z) - z$ constant on each half plane. (where single-valued)

so constant on whole Riemann surface.

Just need to determine constant. Setting $\log(1) = 0$ gives $\phi(0) - 0 = 0$ ✓.

(Fix another choice $2\pi i l$ then $\log(e^z) = z + 2\pi i l$)

same l throughout whole Riemann surface

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Also can define functions along parametrized curves ...

leads to notion of branch point.