Last time : f diff $\iff$ u, v satisfying C-R eqns
+ first partials continuous

Two examples : polynomials , rat'l functions.

generalize them : power series , Laurent series (allowed to have finitely many terms in negative powers n of $(z-z_0)^n$)

and show that

power series $\updownarrow$ holomorphic functions

Laurent series $\updownarrow$ meromorphic functions (to be defined later by generalizing defn of pole)

Today, want to prove that
power series define holomorphic functions , and derivatives given by term-by-term diff. in series.

Big Picture for Course : This week - finish cx. functions

Wed : Exp / Log. Fri : Finish Exp / Log , discuss topology very briefly § 3.1 in Ahlfors

Next week : Start integration theory (skip Möbius transformations/ conformal maps for now.)

Proposition: As a function on $\mathbb{C} \cup \{\infty\}$, $R(z)$ has equal number of zeros, poles (counted with multiplicity) and this equals $\max(\deg P, \deg R)$

pf: Bookkeeping at $z = \infty$ + Fundamental thm of algebra.

More interesting way to form analytic (i.e. holomorphic, differentiable) functions

Use Power Series.

(Read proof in Ahlfors 2.2.1 on completeness of $\mathbb{C}$: sequence converges iff it is Cauchy)

Just as with $\mathbb{R}$, series converge if ~~[illegible]~~ sequence of partial sums converges.

There are stronger notions of convergence:

say $\sum_{n=1}^{\infty} a_n$ converges absolutely if $\sum_{n=1}^{\infty} |a_n|$ converges.

(If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then it converges by Δ-inequality)

Nice fact about absolutely convergent series: can rearrange order of terms without affecting the sum of the series (not so in general)

(Asked to prove this for next week's problem set)

<u>Uniform convergence</u> : Given sequence of functions $\{f_n(x)\}$ for all $x \in E$, some set, if $f_n(x) \to f(x)$ for some x, then given $\epsilon > 0$, $\exists \; N := N(x)$ s.t.

$$|f_n(x) - f(x)| < \epsilon \quad \text{if} \quad n \geq N.$$

If N may be chosen independent of $x \in E$, then we say $\{f_n(x)\}$ converges uniformly to $f(x)$.

Nice non-example from Ahlfors : $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right) x = x$

where $N > |x|/\varepsilon$.

Important property of uniformly convergent sequences: $f_n(x)$ continuous for all n, if $\{f_n(x)\}$ converges uniformly, then the limit is continuous.

— pf — Suppose $f_n(x)$ continuous. $\{f_n\}$ converges uniformly to f. For any $\epsilon > 0$, we can find n s.t. $|f_n(x) - f(x)| < \varepsilon/3$. for <u>all</u> $x \in E$

Since f_n continuous at x_0, $\exists \; \delta > 0$ s.t.

$$|f_n(x) - f_n(x_0)| < \varepsilon/3. \quad \text{if} \quad |x - x_0| < \delta.$$

Thus if $|x - x_0| < \delta$,

$$|f(x) - f(x_0)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)| < \varepsilon$$

//

Natural test for uniform convergence (for series) ← i.e. apply notion of uniform convergence to the sequence of partial sums.

Weierstrass M-test : Let $\sum a_n$ be convergent series ($a_n \geq 0$)

If $|f_n(x)| \leq M \cdot a_n$, M : constant

for all sufficiently large n ,

(for all $x \in E$: set) then f_n converges uniformly on E.

pf : comparison test.

Power series : generally of form $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$, $a_n \in \mathbb{C}$

(* : see note) ("expansion at z_0")

classic example : geometric series $\sum_{n=0}^{\infty} z^n$ since we

have closed form for partial sums: $S_k = 1 + \cdots + z^k$

$= \dfrac{z^{k+1} - 1}{z - 1}$

so converges for $|z| < 1$, diverges for $|z| \geq 1$.

Rmks : (1) Reason for divergence different for $|z| = 1$ vs. $|z| > 1$.

(2) Region of convergence (a disc) is typical , but having closed form for solution is not.

---

* : In practice, we give proofs only for "expansions at $z_0 = 0$" since makes notation a bit less messy. Of course, all statements remain true for general z_0 upon changing vars $z \mapsto z - z_0$.

In fact, most convergence tests for general power series are obtained by comparison with geometric series.

<u>Idea</u>: compare $|a_n z^n|$ with x^n (for $x < 1$, positive fixed real #)

if we can show that, for n suff. large,

$$\underbrace{|a_n z^n|} < x^n < 1, \quad \text{then} \quad \sum_{n=0}^{\infty} a_n z^n \text{ converges absolutely.}$$

$$\sqrt[n]{|a_n|} \cdot |z| < x < 1 \text{ for } n \text{ suff. large.}$$

Find z for which this holds. Naive <u>guess</u>: $|z| \leq \dfrac{1}{\lim\limits_{n\to\infty} \sqrt[n]{|a_n|}}$

But $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|}$ may not exist. Better to use limes superior, "lim sup" for short.

(always exists) write $\overline{\lim}$

$$\overline{\lim_{n\to\infty}}\, x_n = \inf_n \sup_{k \geq n} x_k := l \in [-\infty, \infty]$$

Better to understand defining property: ① For any $\epsilon > 0$, $\exists N$ s.t.

$$x_n < l + \varepsilon \quad \forall\, n \geq N.$$

(This is what we need for pf. convergence/div.)

② For any $\epsilon > 0$, N, $\exists\, n \geq N$ s.t. $x_n > l - \varepsilon$

(i.e. arbitrarily large n for which $x_n > l - \varepsilon$.

(switch roles of inf/sup in definition of $\overline{\lim}$ to get $\underline{\lim}$ "lim inf" and limit exists iff $\overline{\lim}\, x_n = \underline{\lim}\, x_n$)

So let's start over... Let $R = \dfrac{1}{\overline{\lim\limits_{n\to\infty}} \sqrt[n]{a_n}}$.

<u>Theorem</u>: ① $\sum\limits_{n=0}^{\infty} a_n z^n$ converges absolutely for $|z| < R$

and uniformly for $|z| \le \rho < R$.

② The series diverges for $|z| > R$ (because terms grow unbounded)

<u>pf of ①</u>: If $|z| < R$, then $\exists\, x$ with $|z| < x < R$

so $1/x > 1/R \underset{\substack{\uparrow \\ \text{def. of } \overline{\lim}}}{\Rightarrow} \exists\, N$ s.t. $|a_n|^{1/n} < \frac{1}{x} \;\; \forall\, n \ge N$

i.e. $|a_n| < 1/x^n \;\; \forall\, n \ge N$.

so $|a_n z^n| < (\underbrace{|z|/x}_{<1})^n$ for $n \ge N$ so power series converges by comparison test.

For uniform convergence, if $|z| \le \rho < R$,

then pick x s.t. $\rho < x < R$. Then $|a_n z^n| < (\underbrace{\rho/x}_{\substack{\text{absolute const.} \\ \text{indep. of } z}})^n$

so convergence is uniform by Weierstrass M-test.

pf of ② if $|z| > R$ choose x with

$R < x < |z|$, so $1/x < 1/R$. By <u>other</u> defining prop for lim sup, there are ∞-ly many n s.t. $|a_n|^{1/n} > 1/x$.

$\Leftrightarrow \quad |a_n| > 1/x^n$ so $|a_n z^n| > \underbrace{(|z|/x)^n}_{>1}$ for ∞-ly ~~many~~ many n. $\Rightarrow$

terms of series grow without bound.

<u>Theorem</u>, Part 3 : For $|z| < R$, $\sum_{n=0}^{\infty} a_n z^n = f(z)$ defines an analytic function, with derivative given by term-by-term different. + same radius of convergence for $\sum_{n=1}^{\infty} n a_n z^{n-1}$ as for f.

<u>pf</u> : Note that $\sqrt[n]{n|a_n|}$ has same ~~limit~~ lim sup as $\sqrt[n]{|a_n|}$ since $\lim_{n\to\infty} \sqrt[n]{n} = 1$. (see Ahlfors for cute pf. using binomial thm. p. 39)

For $|z| < R$, write

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = S_k(z) + R_k(z)$$

. Then

$\uparrow$ partial sum up to $a_k z^{k-1}$

$\uparrow$ Remainder or "tail" of series

$$\underbrace{\sum_{n=1}^{\infty} n a_n z^{n-1}}_{\text{call this } d(z)} = \lim_{k\to\infty} S_k'(z)$$

$\leftarrow$ since this is just derivative of finite polynomial.

Want to show $f'(z) = d(z)$. Back to the definition of differentiability ...

$$\frac{f(z)-f(z_0)}{z-z_0} - d(z_0) = \left[\frac{s_k(z)-s_k(z_0)}{z-z_0} - s_k'(z_0)\right] \quad (A)$$

$$+ \quad s_k'(z_0) - d(z_0) \quad (B)$$

$$+ \quad \frac{R_k(z) - R_k(z_0)}{z-z_0} \quad (C)$$

(identity true for any k)

Strategy: Show each of three pieces are small. (for k sufficiently large) as $z \to z_0$

(A) : definition of derivative guarantees for any ϵ

$\exists\ \delta$ s.t. $|z-z_0| < \delta \Rightarrow$ (A) $< \varepsilon/3$.

(B) : $\lim_{k\to\infty} s_k'(z_0) = d(z_0)$ so $\exists\ N_B$ s.t. $k \geq N_B$

then (B) $< \varepsilon/3$.

(C) $$\left|\frac{R_k(z)-R_k(z_0)}{z-z_0}\right| \leq \sum_{n=k}^{\infty} |a_n| \left|\underbrace{\frac{z^n - z_0^n}{z-z_0}}_{\underbrace{z^{n-1} + z^{n-2}z_0 + \cdots + z_0^{n-1}}_{n \text{ terms of degree } n-1}}\right|$$

if $|z|, |z_0| < R$ pick x

s.t. $|z|, |z_0| < x < R$

so (C) $\leq \sum_{n=k}^{\infty} |a_n|\, n \cdot x^{n-1} < \infty$. So pick N_c

s.t., for $k \geq N_c$, (C) $< \varepsilon/3$.

so choose $k \geq \max(N_B, N_C)$, and we have shown

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - d(z_0) \right| < \varepsilon \quad \text{whenever} \quad |z - z_0| < \delta .$$

so derivative exists and is equal to $d(z_0)$ for $|z_0| < R$.

corollary: $a_k = f^{(k)}(0) / k!$ (Taylor's formula)

Exploring example of differentiable functions:

Prove differentiability of nice classes of functions directly from the definition — e.g. polynomials, rational functions, power series.

For power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ has radius of convergence R

with $\frac{1}{R} = \limsup_n |a_n|^{1/n}$ s.t. for $|z| < R$, $f(z)$

converges absolutely, differentiable, with derivative given by term-by-term differentiation:

$$f'(z) = \sum_{n=1}^{\infty} n \cdot a_n z^{n-1}$$ (with same radius of conv. R)

Can <u>REPEAT</u> this process, proving power series are inf. diff. for $|z| < R$ with

$$f^{(k)}(z) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n z^{n-k}$$

$$= k!\, a_k + \cdots$$

so $f^{(k)}(0) = k!\, a_k \iff a_k = f^{(k)}(0)/k!$ and hence

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n .$$

<u>CAUTION</u>: Haven't proved anything about arbitrary analytic functions, only those initially defined as a power series.