On Monday, we proved analytic continuation + F.E. of Riemann zeta function. In nicest form,

$$\xi^*(s) \overset{\text{def}}{=} \frac{1}{2} s(1-s)\pi^{-s/2}\Gamma(s/2)\zeta(s), \quad \text{entire function}$$

then $\xi^*(s) = \xi^*(1-s)$.

Today: (0) Comments on analytic cont.

(1) Application to counting primes

(2) What is a modular form? (Preview of next semester)

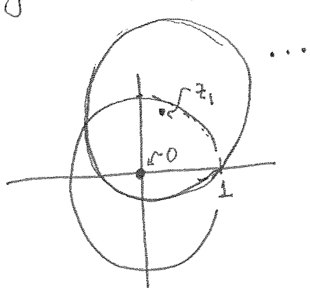
(0): Representations of functions matter w.r.t. analytic continuation (or region of definition)

Almost Trivial Example: $\sum_{k=0}^{\infty} z^k$ has radius of conv. $\dfrac{1}{\overline{\lim\limits_{k\to\infty}} (|z|^k)^{1/k}} = 1$, so defines analytic function on open unit disk.

But we can sum this series, $= \dfrac{1}{1-z}$. This rep'n makes meromorphic continuation to all of $\mathbb{C}$ obvious.

problem: radius of conv. sensitive to poles.

If you love power series (like Weierstrass did) then could try the following:



use repeated Taylor expansions at $z_0 = 0, z_1, \ldots$

Since regions overlap, then resulting analytic function defined by each series must be same.

Or instead of summing series, we can try to find an integral rep'n, as in case of $\zeta(s)$.

3 reasons why Riemann zeta function is important:

① Riemann hypothesis — Conjecture that zeros of $\zeta(s)$ lie on the line $\mathrm{Re}(s) = 1/2$. (axis of symmetry for functional equation)

Progress: No zeros on $\mathrm{Re}(s) \geq 1$.

Proportions of zeros on half line. (i.e. not so close to proof)

② First example of series of form $\displaystyle\sum_{n=1}^{\infty} \frac{a(n)}{n^s}$ $a(n) \in \mathbb{C}$

with analytic cont. + functional eq'n (L-functions)

20th century: turned questions in arithmetic into questions about L-function "Langlands program"

③ Connection to primes

$$\zeta(s) = \prod_{n=1}^{\infty} (1 - p_n^{-s})^{-1} \quad \text{"Euler product"}$$

where we've arranged primes in increasing order:

$$\begin{matrix} 2, & 3, & 5 \\ \| & \| & \| \\ p_1 & p_2 & p_3 \end{matrix}$$

Formally, seems correct by expanding these geometric series

$$(1 - p^{-s})^{-1} = 1 + p^{-s} + p^{-2s} + \cdots$$

Follows from unique factorization.

To check uniform convergence of product consider:

$$\frac{1}{\zeta(s)} = \prod_{n=1}^{\infty} (1 - p_n^{-s})$$

then want $\displaystyle\sum_n |p_n^{-s}|$ to converge uniformly.

True if $\mathrm{Re}(s) \geq \sigma_0 > 1$.

To see that it is equal to $\zeta(s)$ ...

If $Re(s) > 1$, then we may rearrange terms:

$$\zeta(s) \cdot (1 - 2^{-s}) = \sum n^{-s} - \sum (2n)^{-s} = \sum_{m \text{ odd}} m^{-s}$$

Keep going:

$$\zeta(s)(1-2^{-s}) \cdots (1 - p_N^{-s}) = \sum_m m^{-s}$$ where the sum runs over integers without prime factors $2, 3, \ldots, p_N$.

then

$$\lim_{N \to \infty} \zeta(s)((1-2^{-s}) \cdots (1-p_N^{-s}) = 1$$

so $= 1 + p_{N+1}^{-s} + \cdots$

as desired.

(primes can't be finite in number, else this would contradict pole of $s=1$ for $\zeta(s)$)

"Euler's proof" that there are ∞-ly many primes.

Prime number thm: $\pi(x) = \#$ of primes less than $x \sim \dfrac{x}{\log x}$

Step 1: Enough to understand $f(x) = \sum_{n \le x} \Lambda(n)$

where $\Lambda(n)$: von Mangoldt function $= \begin{cases} \log p & \text{if } n = p^k \quad k \geq 1 \\ 0 & \text{else} \end{cases}$

but $$\frac{d}{ds}[-\log \zeta(s)] = -\frac{\zeta'(s)}{\zeta(s)} = \sum_p \sum_{m=1}^{\infty} \frac{\log p}{p^{ms}} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$$

Step 2: Use integral transform

$$I(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^s \frac{ds}{s} = \begin{cases} 0 & \text{if } 0 < x < 1 \\ 1/2 & \text{if } x = 1 \\ 1 & \text{if } x > 1 \end{cases}$$ (for any real $c > 0$)

Inserting $-\zeta'(s)/\zeta(s)$ into the integral transform, we have:

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} -\frac{\zeta'(s)}{\zeta(s)}\, x^s \,\frac{ds}{s} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \sum_{n=1}^{\infty} \Lambda(n) \left(\frac{x}{n}\right)^s \frac{ds}{s}$$

$$= \frac{1}{2\pi i}\sum_{n=1}^{\infty} \Lambda(n) \int_{c-i\infty}^{c+i\infty} \left(\frac{x}{n}\right)^s \frac{ds}{s} = \sum_{n\le x} \Lambda(n) = \psi(x).$$

(take $c>1$ so that Dirichlet series interpretations valid.)

Step 3: Move line of integration leftward, picking up poles

of $\dfrac{\zeta'(s)}{\zeta(s)}\, x^s/s$

Get
$$\psi(x) = x - \sum_{\rho} \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2}\log(1-x^{-2})$$

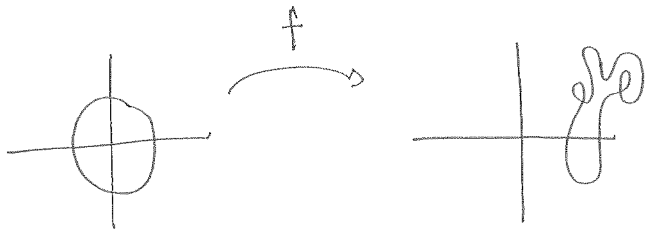
ρ zero of zeta in critical strip

summing residues at trivial zeros.

show this is an error term.

(2) What is a modular form?

A function $f: \mathbb{C} \to \mathbb{C}$ can be thought of as a transformation of space.



E.g. analyze its effect on simple geom. objects like unit circle.

Important class of functions taking circles and lines to circles and lines.

(lines are just circles on $\mathbb{P}^1$ = Riemann sphere)

$$f(z) = \frac{az+b}{cz+d}$$ corresponding to $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $\underbrace{ad-bc=1}_{}$.

guarantees f invertible, in particular.

(See Ahlfors 3.3)

We'll revisit them next semester.

(upper)
$\mathcal{H}$: half-plane

Definition of modular form: $f : \{ z \in \mathbb{C} \mid \operatorname{Im}(z) > 0 \} \to \mathbb{C}$, k : pos. (even) integer.

s.t. $$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$ for all $\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ s.t. } ad-bc=1 \right\}$

$\Theta(z)$ is an example of a modular form (with $k = 1/2$)

$$\Theta(z) = \sum_{n \in \mathbb{Z}} e^{\pi i n^2 z}$$ (used in Riemann's 2nd pf. of A.C. + F.E. of $\zeta(s)$).

Next semester: Study Riemann surfaces (of which $\mathcal{H} / \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \}$ is one) and their modular forms.