On Friday, we proved meromorphic continuation of the zeta function:

<u>Thm</u>: $\zeta(s)$ is meromorphic for $s \in \mathbb{C}$ with only pole a simple pole at $s=1$, with residue 1.

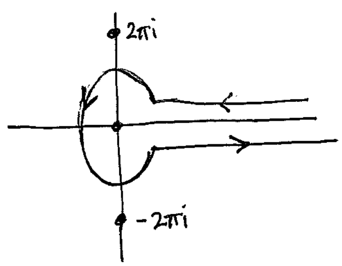
<u>Key idea</u>: Use integral expression for Γ-function

$$\Gamma(s) = \int_0^\infty x^s e^{-x} \frac{dx}{x} \qquad \text{For } \mathrm{Re}(s) > 0 .$$

to prove

$$\zeta(s) = -\frac{\Gamma(1-s)}{2\pi i} \int_C \frac{(-z)^{s-1}}{e^z - 1} \, dz$$

with C:



with $\log(-z)$ defined on $\mathbb{C} \setminus \{$~~pos.~~ non-neg. reals$\}$ $\mathbb{R}_{\geq 0}$.

<u>Thm 2</u>: (Functional equation)

$$\zeta(s) = 2^s \pi^{s-1} \sin\frac{\pi s}{2} \Gamma(1-s) \zeta(1-s)$$

or, defining

$$\zeta^*(s) = \frac{1}{2} (s(s-1)) \pi^{-s/2} \Gamma(s/2) \zeta(s) \qquad (*)$$

then $\zeta^*(s)$ entire and $\zeta^*(s) = \zeta^*(1-s)$.

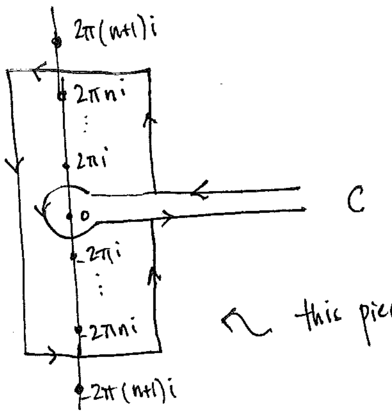
[You might worry about poles of $\Gamma(s/2)$ on RHS of $(*)$. They occur at negative even integers. But $\zeta(s)$ has "trivial zeros" at negative even integers using expansion

$$\frac{1}{e^z - 1} = \frac{1}{z} - \frac{1}{2} + \sum (-1)^{k-1} \frac{B_k}{(2k)!} z^{2k-1} \;]$$

only odds!

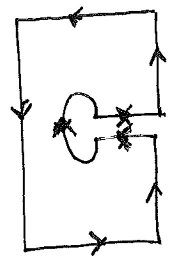
insert into integral over C above, use residue thm ...

Pf of thm 2: Use pair of contours C (as before) and C_n (as below)



this piece is supposed to be a square...

$\Longrightarrow$ $C_n - C$:



"floppy disk contour"

Then by residue thm: $\displaystyle \frac{1}{2\pi i}\int_{C_n - C} \frac{(-z)^{s-1}}{e^z - 1}\, dz = \sum_{m=1}^{n} \left[(-2\pi i m)^{s-1} + (2\pi i m)^{s-1} \right]$

$$= 2 \cdot (2\pi)^{s-1} \sin \frac{\pi s}{2} \sum_{m=1}^{n} m^{s-1}$$

If we take $\mathrm{Re}(s) < 0$, then $\displaystyle \lim_{n \to \infty} \sum_{m=1}^{n} m^{s-1}$ converges to $\zeta(1-s)$.

What about left-hand side? Get expected F.E. if we can show that

$$\int_{C_n - C} \longrightarrow \int_{-C} = -\int_{C} \quad \text{as } n \to \infty.$$

(Assuming $\mathrm{Re}(s) < 0$; thus two sides of FE are meromorphic functions on $\mathbb{C}$ agreeing on $\mathrm{Re}(s) < 0$. Must be same)

Write $C_n = \underbrace{C_n^{sq}}_{\text{part on square}} \cup \underbrace{C_n^{ray}}_{\text{tail outside of square}}$

On C_n^{sq}: $\displaystyle \left| \frac{(-z)^{s-1}}{e^z - 1} \right| < K \cdot n^{\mathrm{Re}(s) - 1}$, $\quad |dz| \le K' \cdot \underbrace{(\text{length of square})}_{\text{order } n}$

So $\displaystyle \left| \int_{C_n^{sq}} \frac{(-z)^{s-1}}{e^z - 1}\, dz \right| < K'' n^{\mathrm{Re}(s)} \to 0$ as $n \to \infty$ if $\mathrm{Re}(s) < 0$.

Finally, on C_n^{ray}, just getting pair of integrals similar to

$$\int_n^\infty \frac{(-z)^{s-1}}{e^z-1}\,dz = \lim_{N\to\infty} \int_n^N \frac{(-z)^{s-1}}{e^z-1}\,dz,$$ and want to take limit as $n\to\infty$.

One way to prove this: $$\left|\int_n^\infty \frac{(-z)^{s}}{e^z-1}\frac{dz}{z}\right| \le \frac{1}{n}\underbrace{\int_n^\infty \frac{z^s}{e^z-1}\,dz}_{\text{convergent}}.$$

or as geometric series since $Re(s)<0$.

Another pf. of analytic continuation: before related $\zeta(s)$ to $\underbrace{\int_0^\infty t^s \frac{1}{e^t-1}\frac{dt}{t}}$ "Mellin transform of $\frac{1}{e^x-1}$"

exponential decay implies well-defined at ∞

but that may have problems near $t=0$ if

$Re(s)$ not >1.

Riemann's other pf. of functional equation:

$$\Theta(z) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2 z}$$

Consider Mellin transform of $\Theta(iy) - \underbrace{\text{const term}}_{=1}$

$$= \int_0^\infty y^s(\Theta(iy)-1)\frac{dy}{y}$$

$$= \int_0^\infty y^s \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} e^{-\pi n^2 y}\frac{dy}{y}$$

Clever bit:

$$= \int_0^1 + \int_1^\infty \qquad = 2\cdot\pi^{-s}\,\Gamma(s)\,\zeta(2s)$$

now take $y\mapsto 1/y$ in first integral ...

Left with: $\displaystyle\int_1^\infty \left[y^s \left(\Theta(iy)-1\right) + y^{-s}\left(\Theta(i/y)-1\right)\right] \frac{dy}{y}$

Wonderful resolution: $\Theta(i/y) \sim y\,\Theta(iy)$

this "functional equation" for Theta function is consequence of 2 facts:

(1) Fourier transform of $e^{-\pi x^2}$ is itself

(2) Poisson summation:

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n)$$

Fourier transform

So in fact obtain merom. continuation + functional equation all at once.

Note: this functional equation is for $\Theta(iy)$, not $\Theta(iy)-1$.

Taking into account the -1 is where poles come in.

Note 2: This transformation for Θ extends to relation for all $z \in \mathbb{C}$. Implies that $\Theta(z)$ is a kind of modular form.