Back to the zeta function:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad s = \sigma + it, \text{ converges absolutely for } \underset{\sigma}{\underset{\shortparallel}{\mathrm{Re}(s)}} > 1.$$

celebrated for connection to number theory:

$$\zeta(s) = \prod_{p:\text{prime}} (1 + p^{-s} + p^{-2s} + \cdots) = \prod_{p:\text{prime}} (1 - p^{-s})^{-1}$$

(clear formally, equivalent to fund. thm. of arithmetic, but make this rigorous by analyzing conv. of partial products for $\mathrm{Re}(s) > 1$)

Get back to this next week. Today explore analytic properties of $\zeta(s)$:

- analytic continuation to merom. function on all of $\mathbb{C}$
- functional equation.

To explore these, need to use Gamma function.

$$\Gamma(s) = \int_0^\infty t^s e^{-t} \frac{dt}{t} \quad \text{for } \mathrm{Re}(s) > 0.$$

Then changing vars $t \mapsto nt$, $n \in \mathbb{Z}_{>0}$ get

$$n^{-s}\Gamma(s) = \int_0^\infty e^{-nt} t^s \frac{dt}{t}.$$

($\frac{dt}{t}$ was Haar meas for $\mathbb{R}_+^\times$)

if $\mathrm{Re}(s) > 1$, then can sum left-hand side over $n \in \mathbb{N}$.

$$\zeta(s)\Gamma(s) = \sum_{n=1}^{\infty} \int_0^\infty e^{-nt} t^s \frac{dt}{t}.$$

Now want to interchange summation + integration.

if so, then $\sum_{n=1}^{\infty} e^{-nt} = \frac{e^{-t}}{1-e^{-t}} = \frac{1}{e^t - 1}$

so we expect $\boxed{\zeta(s)\Gamma(s) = \int_0^\infty t^s \cdot \frac{1}{e^t-1} \frac{dt}{t}}$

To prove rigorously: Pick α, β with $0 < \alpha < \beta < \infty$, so that

$\int_0^\alpha t^{x-1} \frac{1}{e^t-1} dt$ small ($< \epsilon/4$ some ϵ) $(x > 1)$.

$\int_\beta^\infty t^{x-1} \frac{1}{e^t-1} dt$ small ($< \epsilon/4$ some ϵ)

$\Rightarrow \sum_n \int_0^\alpha$ —"— , $\sum_n \int_\beta^\infty$ —"— small

then use uniform convergence of $\sum e^{-nt}$ to $\frac{1}{e^t-1}$ on $[\alpha, \beta]$.

Thm: For $\mathrm{Re}(s) > 1$,

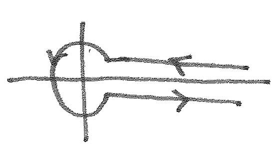
$$\zeta(s) = -\frac{\Gamma(1-s)}{2\pi i} \int_C \frac{(-z)^{s-1}\, dz}{e^z - 1}$$

where C is "infinite keyhole"

$-z^{s-1} := e^{(s-1)\log(-z)}$

log branch has cut along (of $(-z)$) positive real axis

$-\pi < \mathrm{Im}(\log(-z)) < \pi$

C: 

To evaluate: Only use Cauchy's thm to say that as we shrink contour, the value of the integral remains constant.

Usually estimates demonstrate integral over circular part $\to 0$ as its radius $\to 0$

$$(-z)^{s-1} = x^{s-1} e^{-(s-1)\pi i} \quad \text{on upper edge}$$
$$= x^{s-1} e^{(s-1)\pi i} \quad \text{on lower edge.}$$

$$\Rightarrow \int_C \frac{(-z)^{s-1}}{e^z - 1} dz = \Big[\underbrace{e^{(s-1)\pi i} - e^{-(s-1)\pi i}}_{2i \sin (s-1)\pi} \Big] \zeta(s) \Gamma(s)$$

do a little cleaning up: $\sin (s-1)\pi = -\sin s\pi$, $\Gamma(s)\Gamma(1-s) = \pi / \sin s\pi$.

so done.

Corollary: $\zeta(s)$ can be extended to a meromorphic function $\forall s \in \mathbb{C}$ whose only pole is a simple pole at $s=1$ with residue 1.

pf: $\displaystyle\int_C \frac{(-z)^{s-1}}{e^z-1} dz$ defines entire function (integral well-defined for all s since growth of e^z faster than z^s for any fixed s.)

then differentiate under integral sign

poles come from $\Gamma(1-s)$ at $s = 1, 2, \ldots$

but we know $\zeta(s)$ defined for $s = \sigma + it$ with $\sigma > 1$,

so $s = 2, 3, \dots$ can't be poles. (In fact they cancel 0's from integration)

Only pole unaccounted for is $s = 1$.

Here the integral is $\displaystyle \frac{1}{2\pi i} \int_C \frac{dz}{e^z - 1} = 1$ (residue thm.)

from product expansion, $\displaystyle \Gamma(z) = e^{-\gamma z} z^{-1} \cdot \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n}$

then residue at pole at 0 comes from $\frac{1}{z}$, residue $= 1$

Same as that for $-\Gamma(1-s)$.

Conclusion: $\operatorname{res}_{s=1} \zeta(s) = 1$.

Functional equation:

$$\zeta(s) = 2^s \pi^{s-1} \sin \frac{\pi s}{2} \, \Gamma(1-s) \, \zeta(1-s)$$

or more symmetrically:

setting $\displaystyle \xi^*(s) = \frac{1}{2} s \cdot (s-1) \pi^{-s/2} \Gamma(s/2) \zeta(s)$

then $\xi^*(s)$ is entire and $\xi^*(s) = \xi^*(1-s)$.

pf: Another contour integration.