Explore properties of Riemann zeta function (analytic continuation / functional equation) and their application (via contour integration) to the prime number theorem.

Zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} : \mathbb{C} \to \mathbb{C}$.

This series converges for $\mathrm{Re}(s) > 1$ and uniformly for all s with $\mathrm{Re}(s) \geq \sigma_0 > 1$ for any fixed real # σ_0.

So represents analytic function of s for $\mathrm{Re}(s) > 1$.

(Weierstrass' Thm)
General thm: f_n analytic in Ω_n

$\{f_n(z)\} \to f$ in Ω: region for which each $z \in \Omega$ has $z \in \Omega_n$ $\forall\, n \geq N_z$

uniformly on every compact subset of Ω.

Then f is analytic in Ω

(and $f_n' \to f'$ uniformly on every compact subset of Ω)

One pf: Cauchy's integral formula: Given any $a \in \Omega$, Pick closed disk contained in Ω: $\overline{B(a,r)}$.

Then $f_n(z) = \frac{1}{2\pi i} \int_{C(a,r)} \frac{f_n(\xi)\, d\xi}{\xi - z}$ for $z \in B(a,r)$. Take limit as $n \to \infty$ uniform convergence gives

$$f(z) = \frac{1}{2\pi i} \int_{C(a,r)} \frac{f(\xi)\, d\xi}{\xi - z}$$

Similar formula for derivatives via integral formula.

same fact we used for existence of power series: $\longrightarrow$ (need uniform convergence to guarantee that limit function is continuous, hence gives well-defined integral.)

F continuous, $F = \lim_n F_n$
then $\int F\, dz = \lim_n \int F_n\, dz$. cf. Ahlfors p. 36

For analytic continuation: Better to consider

$$\xi^*(s) = \pi^{-s/2} \underbrace{\Gamma(s/2)}_{\text{Gamma function}} \xi(s).$$

Then $\xi^*(s)$ extends to merom. function of $\mathbb{C}$ w/ simple pole at $s=1$, satisfying the functional equation:

$$\xi^*(s) = \xi^*(1-s).$$

What is Γ-function? Why natural?

Answer comes from study of (meromorphic) functions with prescribed zeros or poles.

Easy enough when # of zeros / poles finite. What if infinite? (but of course isolated, which is required by definition)

Given f merom. on Ω.

One idea: a_i : pole, then let

$P_i\left(\frac{1}{z-a_i}\right)$ be singular part of Laurent exp. of f at $z = a_i$. (P_i : b/c these are polynomials (with no const. term))

Consider $f(z) = \sum_{i:\, a_i \text{ pole}} P_i\left(\frac{1}{z-a_i}\right) + \underbrace{g(z)}_{\text{analytic in } \Omega}$.

Issue: with ∞-ly many poles $\sum_i P_i\left(\frac{1}{z-a_i}\right)$ may not converge.

Thm (Mittag-Leffler) $\{a_i\}$: cx. numbers $\lim_{i\to\infty} |a_i| = \infty$.

and P_i : polynomials w/o const. term.

Then there are functions f, meromorphic on $\mathbb{C}$, with poles at a_i, singular parts P_i

All such functions can be written in form:

$$f(z) = \sum_i \left[P_i\left(\frac{1}{z-a_i}\right) - p_i(z) \right] + g(z) \qquad (*)$$

p_i : suitably chosen polys., $g(z)$: entire.

<u>pf</u>: w.l.o.g., assume none of a_i are 0. (else shift function)

Then $P_i\left(\frac{1}{z-a_i}\right)$ is analytic for $|z| < |a_i|$, so has Taylor series expansion at origin. Then let $p_i(z)$ be Taylor polynomial at 0 with ~~[illegible]~~ degree n_i (to be chosen/... specified)

By Cauchy's inequality: if max of $\left|P_i\left(\frac{1}{z-a_i}\right) - p_i(z)\right|$ on $|z| \le |a_i|/2$ is given by M_i then

the remainder
$$\underbrace{P_i\left(\frac{1}{z-a_i}\right) - p_i(z)}_{F(z)} = \left(\frac{1}{2\pi i}\int_C \frac{F(\xi)}{(\xi)^{n_i+1}(\xi - z)}\,d\xi\right)(z-0)^{n+1}$$

so
$$\left|P_i\left(\frac{1}{z-a_i}\right) - p_i(z)\right| \le \frac{1}{2\pi} M_i \left(\frac{2|z|}{|a_i|}\right)^{n_i+1} \cdot \frac{1}{|a_i|/4} \underbrace{\int_C |d\xi|}_{2\pi\cdot\frac{|a_i|}{2}}$$

where $C = C(0, |a_i|/2)$ and we restrict z to $B(0, |a_i|/4)$.

If we choose n_i large enough, e.g. so that $2^{n_i} \ge M_i \cdot 2^i$, then the RHS of the above inequality will be $2M_i \cdot \left(\frac{2|z|}{|a_i|}\right)^{n_i+1}$ and $\frac{|z|}{|a_i|} \le 1/4$

so $\le 2 \cdot M_i\, 2^{-(n_i+1)} \le 2^{-i}$ for $|z| \le |a_i|/4$.

<u>Claim</u>: series in RHS of $(*)$ in theorem converges uniformly on any (closed) disk $B(0,R)$ except at poles. and hence represents meromorphic function

Write sum as $h(z) :=$

$$\sum_i P_i\left(\frac{1}{z-a_i}\right) - p_i(z) = \sum_{i:\,\frac{|a_i|}{4} \le R} P_i\left(\frac{1}{z-a_i}\right) - p_i(z) + \sum_{i:\,\frac{|a_i|}{4} > R} P_i\left(\frac{1}{z-a_i}\right) - p_i(z)$$

finite sum. $p_i(z)$ poly.

holom. for $|z| \le R$ since thus our estimate applies this proves result since any other f meromorphic with these properties will have $f(z)-h(z)$ holomorphic.

Example: $\dfrac{\pi^2}{\sin^2 \pi z}$, which has double poles at all integers (and nowhere else in $\mathbb{C}$ — estimate $|\sin z| \geq \max\left(\frac{e^{y}-e^{-y}}{2}\right.$ and $\left.\left(\frac{e^{-y}-e^{y}}{2}\right)\right)$ using reverse triangle inequality.)

singular parts: at $z=0$: of the form $\dfrac{c_{-2}}{z^2} + \dfrac{c_{-1}}{z} + \phi(z)$.

multiply by z^2, take limit as $z \to 0$,

$c_{-2} = 1$, and $\sin^2 \pi z$ even, so $c_{-1} = 0$.

Then by periodicity, ~~[illegible]~~ $\sin^2 \pi(z-n) = \sin^2 \pi z$, so the singular part is the same at all integers: $\dfrac{1}{(z-n)^2}$.

Thus Mittag-Leffler's thm tells us that

$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} + g(z)$$

since $\displaystyle\sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$ is convergent for any $z \neq n$, integer and absolutely conv. on any compact set.

So we don't need to correct with Taylor polys.

What is $g(z)$? Claim: $\equiv 0$.

Idea: $g(z)$ periodic since other two functions in identity are periodic, of period 1. Suffices to show g is bounded on any strip of width 1. Then Liouville's thm implies g is zero (since g bounded, entire)

show g bounded in strip by showing $g(z) \to 0$ uniformly on strip.

Follows since $|\sin \pi z|^2 = \cosh^2 \pi y - \cos^2 \pi x$

$\cosh y = \dfrac{e^{y}+e^{-y}}{2} \to \infty$ as $|y| \to \infty$.

so $\dfrac{1}{\sin^2 \pi z} \to 0$ as $|y| \to \infty$ uniformly.