Fourier analysis: exploit symmetry.

Working over space X on which group G acts, then try to decompose arbitrary functions on X as sums / integrals of basic functions

those which transform nicely under gp. action.

Examples: $\mathbb{R}$, $\mathbb{R}/\mathbb{Z}$: functions on $\mathbb{R}/\mathbb{Z}$ $\updownarrow$ periodic functions on $\mathbb{R}$ (of period 1)

$\mathbb{R}/\mathbb{Z} = T$: torus

gps under addition, so act on themselves by translation.

what are gp. representations? All characters since G abelian.

irred. repns are

basic functions are characters: $\phi(x+y) = \phi(x)\cdot\phi(y)$

from requirement that $f(x+y) = \phi(y)\cdot f(x)$ so $f(0+y) = \phi(y) f(0)$.

Really simple example : $\mathbb{Z}/2\mathbb{Z} \circlearrowright \mathbb{R}$ by $\mathbb{R} \to \mathbb{R}$
$1 : x \mapsto x$
$-1 : x \mapsto -x$

even, odd big classes of functions.

characters of $\mathbb{Z}/2\mathbb{Z}$: triv. : $x \mapsto 1$ $(\simeq \mathbb{Z}/2\mathbb{Z})$
sgn. $x \mapsto \mathrm{sgn}(x)$

$f(\pm 1 \cdot x) = \mathrm{triv}(\pm 1) f(x)$: even.

$f(\pm 1 \cdot x) = \mathrm{sgn}(\pm 1) f(x)$: odd

Write $f = f_{even} + f_{odd}$

use orthogonality of characters

$\chi \neq \mathrm{triv} : \sum_{a \in G} \chi(a) = 0$

Not too hard to prove that:

characters of $\mathbb{R}$ (or even $\mathbb{R}^n$) : $x \mapsto e^{2\pi i \xi x}$ $\quad \xi \in \mathbb{R}$ (or $\mathbb{R}^n$)

where ξx is $\xi \cdot x$.

characters of $\underbrace{\mathbb{R}/\mathbb{Z}}_{T}$: $x \mapsto e^{2\pi i \xi x}$ $\quad \xi \in \mathbb{Z}$

(or $\mathbb{Z}^n$ for T^n)

characters of $\mathbb{R}/\mathbb{Z} \simeq \mathbb{Z}$. (not itself, countable collection)

forms orthonormal basis of $L^2(T)$ (or T^n)

Given f, express it in form

$$f(x) = \sum_{n \in \mathbb{Z}} a(n) e^{2\pi i n \cdot x}$$

(converges in $L^2(T)$)

where $a(n) = \int_T f(x) e^{-2\pi i n x}\, dx =: \hat{f}(n)$ Fourier transform of f.

more common notation.

over $\mathbb{R}$ (or $\mathbb{R}^n$) slightly more complicated

$$f(x) = \int_{\mathbb{R}} \hat{f}(\xi) e^{2\pi i x \cdot \xi}\, d\xi$$

where $$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \cdot \xi}\, dx$$

if $f \in L^1$, then integral for $\hat{f}$ converges, but no map back, unless $\hat{f}$ is L^1 (not guaranteed). f needs to suitably restricted.

why <u>useful</u>: $f, \hat{f}$ have very different properties. * (see next page)

Fourier inversion <u>thm</u>: $\check{f}(x) := \hat{f}(-x) = \int_{\mathbb{R}} f(\xi) e^{2\pi i x \cdot \xi} d\xi$

then $(\hat{f})^{\vee} = (\check{f})^{\wedge} = f$.

(if $f, \hat{f} \in L^1$)

(might think this is Fubini's thm, but more subtle.
Standard pf using convolution, approximations to identity
and one important fact:

Calculate Fourier transform of $f(x) = \underbrace{e}_{\text{Gaussian distribution}}{}^{-\pi x^2}$

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$ (cute trick of ~~[illegible]~~ squaring integral $e^{-\pi(x^2+y^2)} dx\,dy$ using polar coordinates. $dx\,dy = r\,dr\,d\theta$)

$$\hat{f}(\eta) = \int_{\mathbb{R}} f(x) e^{-2\pi i \eta x} dx$$

complete the square

use Cauchy's thm

claim $= 1$

$$= \int_{\mathbb{R}} e^{-\pi x^2 - 2\pi i \eta x} dx$$

$$= \int_{\mathbb{R}} e^{-\pi(x+i\eta)^2} \underbrace{e^{-\pi\eta^2}}_{f(\eta)} dx = f(\eta) \int_{\mathbb{R}} e^{-\pi(x+i\eta)^2} dx$$

Application to PDEs:

$$\Delta := \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \quad \text{on } \mathbb{R}^n.$$

$$\Delta(f(x+y)) = (\Delta f)(x+y)$$

so $e^{2\pi i x \cdot \xi}$ are eigenfunctions of Δ. (also check this directly...)

nice properties like: $\widehat{\Delta f}(\xi) = -4\pi^2 |\xi|^2 \hat{f}(\xi)$

i.e. fourier transform turns Laplacian Δ into multiplication.

(see Folland § 8.7 for details)

* fourier transforms of functions:

$$\int_{-\infty}^{\infty} \frac{f}{g} e^{i*} d*$$ was computing fourier transform of f/g.

(need to replace $*$ with $2\pi \xi x$
$\xi \in \mathbb{R}$
evaluate result as function of ξ)

e.g. $\frac{f}{g} = \frac{1}{x^n}$

then transform: $c \cdot \xi^{n-1} \cdot \mathrm{sgn}(\xi)$

basic principle: f concentrated, then $\hat{f}$ spread out and vice-versa

$f = 1$ then $\hat{f} = \delta$ and vice-versa.