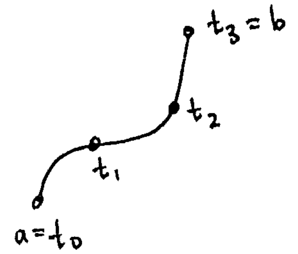
Whenever we subdivide a smooth curve:

$\gamma(t)$ with $t \in [a,b]$ with $t_1 < \dots < t_{k-1} \in [\overset{t_0}{\overset{\|}{a}}, \overset{t_k}{\overset{\|}{b}}]$

<u>write</u>: $\gamma(t) = \gamma_1(t) + \gamma_2(t) + \cdots + \gamma_k(t)$ with $\gamma_i(t)$ having $t \in [t_{i-1}, t_i]$



Then $\displaystyle\int_\gamma f\,dz = \int_{\gamma_1+\gamma_2+\gamma_3} f\,dz = \sum_i \int_{\gamma_i} f\,dz$

More generally, define integral over arbitrary formal sum of curves by same equality, even when they don't correspond to a subdivision of a curve:

$$\int_{\sum \gamma_i} f\,dz \stackrel{\text{def}}{=} \sum_i \int_{\gamma_i} f\,dz. \qquad \text{(finite sums)}$$

This formal sum is called a "chain". Place equivalence relation on chains

by $\sum \gamma_i \sim \sum \gamma_j'$ if $\displaystyle\int_{\sum_i \gamma_i} f\,dz = \int_{\sum_j \gamma_j'} f\,dz$ for $\underbrace{\text{all}\ \ f}_{\text{various notions here.}}$.

all continuous functions on Ω containing both chains.

A chain which is representable as a sum of closed curves is called a <u>cycle</u>

(just check endpts. See if they occur in pairs)

Want to generalize Cauchy's theorem to open, connected sets "w/o holes."

Such regions are called "simply connected".

Proper definition of "simply connected"?

Usually a connected open set Ω is said to be "simply connected" if every closed curve is homotopically trivial.

Recall that a homotopy between two closed curves is a continuous deformation from one to the other:

Suppose $\gamma_1, \gamma_2 : [0,1] \to \Omega$ closed curves. Say γ_1 is homotopic to γ_2 if $\exists$ continuous function $\Gamma : [0,1] \times [0,1] \to \Omega$ s.t.
(s,t)

s.t. $\Gamma(s,0) = \gamma_1(s)$, $\Gamma(s,1) = \gamma_2(s)$ $s \in [0,1]$

with $\Gamma(0,t) = \Gamma(1,t)$ $\forall\, t \in [0,1]$

So ~~anyone~~ for any given $t := t_0$, $\Gamma(s,t_0)$ is closed curve, and as we vary t, $\Gamma(s,t)$ continuously deforms between γ_1, γ_2

Being homotopic to a point, i.e. "homotopically trivial" just means that γ_1 homotopic to constant curve: $\gamma_2(s) = z_0$. (z_0 constant $\in \mathbb{C}$.)

Exercise: Show that this is well-defined. I.e. if γ homotopic to z_0, then γ homotopic to z_1 for any other point in Ω. in Ω

This is not Ahlfors' approach. He begins with a definition valid only in the plane.

Definition: Ω open, connected is simply-connected if its complement in $\mathbb{C} \cup \{\infty\}$ is connected.

As he notes, important to include $\{\infty\}$. Otherwise horizontal strip would not be simply connected. Favors this definition b/c it conforms to our intuition about "having no holes"

Immediately, he proves the following equivalence:

Thm: Ω : open, conn. is simply connected $\iff$ $n(\gamma, a) = 0$
(winding # of a w.r.t. γ)
for all cycles $\gamma \subseteq \Omega$
all $a \notin \Omega$.

Pf: ($\Rightarrow$) Given cycle $\gamma \subseteq \Omega$, then if complement of Ω connected, it must be contained in one of the open connected components determined by γ. Must be unbounded region since $\{\infty\}$ is in both it and complement of Ω. $\Rightarrow$ $n(\gamma, a) = 0$ for all a in complement of Ω

($\Leftarrow$) (contrapositive) Suppose Ω has complement $= A \cup B$, disjoint closed sets in $\mathbb{C} \cup \{\infty\}$
One contains $\{\infty\}$, say B, so then A bounded.

Cover the plane with squares of size δ (i.e. tile) $\{Q_j\}$ where, given any $a \in \mathbb{C}$, a is a center for one of the squares.

δ sufficiently small so that no square simultaneously contains points of A, B.

Consider $$\gamma = \sum_{\substack{j: \\ Q_j \cap A \neq \phi}} \partial Q_j$$

Then $n(\gamma, a) = 1$ since a belongs to unique Q_j.

Easy to see $\gamma \cap B, \gamma \cap A = \phi$ so $\gamma \subseteq \Omega$.

New definition: A cycle $\gamma \subseteq \Omega$ is said to be homologous to zero in Ω if $n(\gamma, a) = 0$ for all a in the complement of Ω. Write $\gamma \sim 0$

General statement of Cauchy's thm:

If $f(z)$ analytic in Ω, then $\int_\gamma f(z)\,dz = 0$ for every cycle γ which is homologous to zero in Ω.

Corollary: If f analytic in simply connected region Ω, then for all cycles $\gamma \subseteq \Omega$, $\int_\gamma f(z)\,dz = 0$.

Corollary 2: In simply connected region Ω, every analytic function f is realized as the derivative of an analytic function F.

Corollary 3: If f analytic, $\neq 0$ on Ω, simply connected, then $\log f(z)$, $\sqrt[n]{f(z)}$ have (single-valued) branches defined for all Ω.

pf: Assumption implies $f'(z)/f(z)$ analytic in Ω. Applying corollary 2 to this function, then $\exists$ $F(z)$ on Ω analytic s.t. $F'(z) = f'(z)/f(z)$.

But then $\frac{d}{dz}\left(f(z)e^{-F(z)}\right) = f'(z)e^{-F(z)} + f(z)\cdot\left(-f'(z)/f(z)\right)e^{-F(z)}$

$= 0$ on Ω, so must be constant.

(reasonable to guess since $F(z)$ "=" $\log f(z)$.)

Then $e^{F(z)}\cdot c = f(z)$. Eval. at any $z_0 \in \Omega$. (Pick value for $\log f(z_0)$)

$e^{F(z_0)}\cdot c = f(z_0)$. Take logs. $F(z_0) + \log c = \log f(z_0)$

or $c = e^{\log f(z_0) - F(z_0)}$

For $\sqrt[n]{f}$ note this is $\exp(\frac{1}{n}\log f$

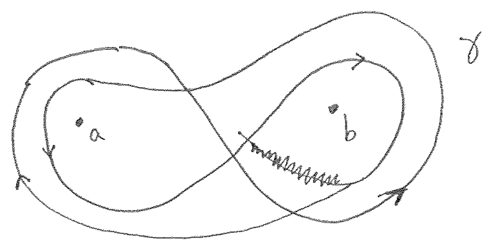
Set $\log f(z) = F(z) - F(z_0) + \log f(z_0)$. Analytic, single-valued.

Two topological properties: homotopy, homology.
$\sim$ $\approx$

Write $\gamma \sim 0$ if γ is homotopic to constant curve. This homotopy is occurring inside open set Ω.

Write $\gamma \approx 0$ if $n(\gamma, w) = 0$ for all $w \in \mathbb{C} \setminus \Omega$.

$\gamma \sim 0 \Rightarrow \gamma \approx 0$ but here's picture which shows reverse implication is not true: Let $\Omega = \mathbb{C} \setminus \{a, b\}$. (So $\mathbb{C} \setminus \Omega = \{a, b\}$)



$\gamma \not\sim 0$
(isn't a proof, but picture is suggestive...)

But $n(\gamma, a) = n(\gamma, b) = +1 - 1 = 0$.

<u>Cauchy's Thm</u> (Homotopy version): $f : \Omega \to \mathbb{C}$ analytic. $\gamma \subseteq \Omega$, smooth closed curve. $\gamma \sim 0$.

Then
$$\int_\gamma f \, dz = 0.$$

<u>Two proofs</u>: Since $\gamma \sim 0$, write homotopy $\Gamma : \overset{s}{[0,1]} \times \overset{t}{[0,1]} \to \Omega$

$\Gamma(s, 1) = \gamma$

$\Gamma(s, 0) = z_0$: const. curve for some $z_0 \in \Omega$

Call $\gamma_t(s) = \Gamma(s, t)$ $s \in [0,1]$
(closed curve)

Define $h(t) = n(\gamma_t ; z)$, $z \in \mathbb{C} \setminus \Omega$. (Need to assume γ_t smooth)

If we can show h is continuous on $t \in [0,1]$, then $h \equiv 0$ since it is discretely valued, ($\in 2\pi i \mathbb{Z}$) and $h(0) = n(z_0 ; z) = 0$.

$\Rightarrow n(\gamma ; z) = 0$. Done by prev. version.