Today: Geometry of complex numbers.

Last time, noted that the distance function was given by:

$$|z|^2 = \bar{z} z = x^2 + y^2 \quad \text{if} \quad z = x+iy$$

so distance $d(z,w) := |z-w|$

where $$|z-w|^2 = |z|^2 + |w|^2 - 2\,\mathrm{Re}\, z\bar{w}$$

Similarly, $$|z+w|^2 = |z|^2 + |w|^2 + 2\,\mathrm{Re}\, z\bar{w}. \qquad (*)$$

Want to confirm we have metric. i.e. $d(z,w)$ satisfies

(1) $d(z,w) \geq 0$ and $d(z,w) = 0$ iff $z = w$.

(2) $d(z,w) = d(w,z) \quad \forall\, z,w \in \mathbb{C}$

(3) $d(z,w) \leq d(z,u) + d(u,z)$ (triangle inequality).

((3) is intuitively clear geometrically, since $d(z,w)$ is distance between points z,w in $\mathbb{R}^2$. But should verify this analytically or algebraically.)

pf of (3): Use $(*)$, noting that

$$|\mathrm{Re}(z)| \leq |z| \quad \text{and} \quad |z\bar{w}| = |z||\bar{w}| = |z||w|$$

so indeed $$|z+w|^2 \leq |z|^2 + |w|^2 + 2|z||w|. \checkmark$$

Note used $\mathrm{Re}(z\bar{w}) \leq |z\bar{w}|$ so get equality if and only if $z\bar{w}$ is non-neg. real.

$\Leftrightarrow$ z/w non-neg real.

so $(\mathbb{C}, |\cdot|)$ is metric space (just $\mathbb{R}^2$ with Euclidean norm)

Two other inequalities of note:

define open ball + open set

(1) $|z-w| \geq \big||z|-|w|\big|$

pf: $|z| = |(z-w)+w| \underset{\uparrow\ \Delta\text{-ineq.}}{\leq} |z-w| + |w| \Rightarrow |z|-|w| \leq |z-w|$

Play same game with $|w|$ to get $|w|-|z| \leq |w-z| = |z-w|$.

(2) Cauchy's inequality: $\left|\sum_{i=1}^{n} z_i w_i\right|^2 \leq \sum_{i=1}^{n} |z_i|^2 \sum_{i=1}^{n} |w_i|^2$

pf: Introduce cx. parameter λ

$$\underbrace{\sum_{i=1}^{n} |z_i - \lambda \bar{w}_i|^2}_{\text{non-negative.}} = \underbrace{\text{EXPAND VIA } (*)}_{\text{optimize this expression as function of } \lambda.}.$$

Back to metric spaces, consider new model for $\mathbb{C} \cup \{\infty\}$.

Often functions take the value ∞, it would be nice to be able to pose questions about continuity at ∞, etc. and to place this value on equal footing with complex numbers.

Not a field, but rather set with additional rules:

$z \cdot \infty = \infty$ $\qquad$ $z/0 = \infty$ if $z \neq 0$

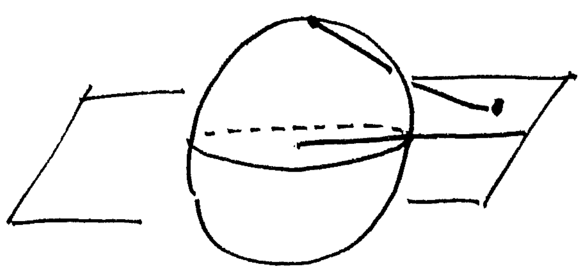
$z + \infty = \infty$ $\qquad$ $z/\infty = 0$ (if $z \neq \infty$)

$\infty \cdot \infty = \infty$

Geometric model for $\mathbb{C} \cup \{\infty\}$ — unit sphere in $\mathbb{R}^3$ (Riemann Sphere)

$$S: \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 + x_3^2 = 1\}$$

and identify $\mathbb{C}$ with $(x_1, x_2, 0)$.



Idea: points $\overset{z}{(x_1, x_2, 0)}$ in $\mathbb{C}$ correspond to $Z = (a_1, a_2, a_3)$ on S where Z lies on line connecting $z = (x_1, x_2, 0)$ and $(0,0,1)$: north pole of S

More explicitly. (because we want algebraic/analytic pfr. not just geometric intuition)

line connecting $z = (x_1, x_2, 0)$ and $(0,0,1)$ given by parametric equation:

$$\{((1-t)x_1, (1-t)x_2, t) \mid t \in \mathbb{R}\} \quad (**)$$

$$\shortparallel$$

$$(1-t)(x_1, x_2, 0) + t \cdot (0,0,1)$$

want point on line s.t.

$$\underbrace{(1-t)^2(x_1^2 + x_2^2)} + t^2 = 1 \qquad \text{(i.e. on } S\text{)}$$

$$(1-t)^2 |z|^2 = 1 - t^2$$

Solve for t. Since $z \neq \infty$, $t \neq 1$, cancel $(1-t)$'s to get

$$t = \frac{|z|^2 - 1}{|z|^2 + 1}$$

If you substitute back into $(**)$, then

$$Z = (a_1, a_2, a_3) = \left(\frac{2x_1}{|z|^2+1}, \frac{2x_2}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1} \right) \qquad (***)$$

$x_1 = \mathrm{Re}(z)$, $x_2 = \mathrm{Im}(z)$

In reverse direction, given

$Z = (a_1, a_2, a_3)$, then $t = a_3$ and can again use $(**)$

to get $x_1 = \dfrac{a_1}{1-a_3}$, $x_2 = \dfrac{a_2}{1-a_3}$. So indeed we have a bijection btw. $\mathbb{C} \cup \{\infty\}$ and S.

As our distance function $d(z,z')$, we take Euclidean distance in $\mathbb{R}^3$ from points Z, Z' on S, corresponding to (lower case) $z, z' \in \mathbb{C} \cup \{\infty\}$.

"chordal distance" (defines same open sets as usual distance)

Again, explicitly,

$$d(z,z')^2 = (a_1 - a_1')^2 + (a_2 - a_2')^2 + (a_3 - a_3')^2$$

$$= 2 - 2(a_1 a_1' + a_2 a_2' + a_3 a_3')$$

since Z, Z' on S

Now substitute $(***)$, to get:

$$d(z,z') = \frac{2|z-z'|}{\left((1+|z|^2)(1+|z'|^2)\right)^{1/2}} \qquad z, z' \in \mathbb{C}$$

$$d(z,\infty) = \frac{2}{(1+|z|^2)^{1/2}}$$

__Note__: this distance function satisfies triangle inequality since we know same is true of Euclidean distance in $\mathbb{R}^3$.

Advantage of Riemann sphere: compact

So far, have rectangular coordinate model (addition corresponds to vector addition)

+ Riemann sphere (allows us to consider $\{\infty\}$ as value taken by function)

Missing a good model for multiplication : Polar coordinates.

$$z = x + iy = r(\cos\theta + i\sin\theta) \qquad r \geq 0$$

(representation not unique since (r, θ) and $(r, \theta + 2\pi k)$ $k \in \mathbb{Z}$ represent same point.)

with $|z| = r$ since $\cos\theta + i\sin\theta$ on complex unit circle.

MAGIC : $$z_1 z_2 = r_1 r_2 \Big[\underbrace{(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2)}_{\cos(\theta_1+\theta_2)} + i\underbrace{(\cos\theta_1\sin\theta_2 + \cos\theta_2\sin\theta_1)}_{\sin(\theta_1+\theta_2)}$$

Call θ the "argument" of z, and write $\arg(z) = \theta$.

Then $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$.

Gives geometric interpretation as homothety : rotation (by $\arg(z)$) + dilation (by $|z|$) of mult. by z

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Polar representation gives simple way to solve monomial equations

$$z^n = a \qquad \text{write} \qquad a = \rho(\cos\phi + i\sin\phi)$$
$$z = r(\cos\theta + i\sin\theta)$$
$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

so $r = \rho^{1/n}$, $\theta = \phi/n + 2\pi k/n$

(gives n distinct solutions)