Play around with maximum principle:

On a disk D of radius R: If we know $|f(z)| \le M$ for $z \in \partial D$

then either $f(z) = M$ on D

or else $\exists\, z_0$ s.t. $|f(z_0)| < M$.

Example: D: unit disk. $M = 1$. $f(0) = 0$ for convenience.

Then $|f(z)| \le |z|$, $|f'(0)| \le 1$.

Moreover, if $|f(z)| = |z|$ for some $z \ne 0$ or $|f'(0)| = 1$ then

$f(z) = cz$ with for some c $|c| = 1$.

Idea: Apply max. principle to $\frac{f(z)}{z}$ (removable singularity at $z = 0$)

so define value at 0 to be $f'(0)$.

for any z on circle $|z| = r < 1$,

this function has modulus $\le \frac{1}{r}$, so must be $\le \frac{1}{r}$ on interior.

as $r \to 1$, get $|f(z)/z| \le 1$. (if equality holds, then $\frac{f(z)}{z}$ must be constant.)

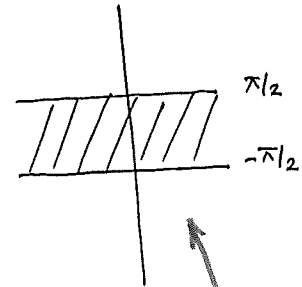
Can sup this up for circles centered at z_0 with $f(z_0) = w_0$, radius R: messier.

See (36) in Ahlfors.

Useful in homework.

Aside: What can be done using only Cauchy's inequality?

Maximum modulus principle applies to bounded domains, what can be said about functions on arbitrary region Ω if we know values of function on $\partial\Omega$. (say $|f(z)| \le M$ on $\partial\Omega$)



Example: e^{e^z} on horizontal strip : $\{ z \mid |\mathrm{Im}(z)| < \pi/2 \}$

On boundary, $\left| e^{e^z} \right| = \left| e^{e^x(\cos y + i \sin y)} \right| = e^{e^x \cos y}$

But $\cos y = 0$ for $y = \pm \pi/2$ so $\left| e^{e^z} \right| = 1$ on boundary.

However $e^{e^z} \longrightarrow \infty$ as $\mathrm{Re}(z) \longrightarrow \infty$.

better to draw only half strip given the coming mods.

Possible modifications : Suppose that on half-strip w/ $\mathrm{Re}(z) \geq 0$, then

$$|f(z)| \ll e^{e^{c \cdot \mathrm{Re}(z)}} \quad (0 < c < 1)$$

(i.e. for $|z| \gg 0$, we have $|f(z)| < \underbrace{K_f}_{\text{const. dep. on } f} e^{e^{c \cdot \mathrm{Re}(z)}}$)

then can use maximum principle to prove in fact $|f(z)| \le M$ $\forall$ z in half-strip.

See Garrett vignette for proof.

Other possible modifications :

For unbounded domains Ω, $\partial_\infty \Omega :=$ finite boundary $\partial\Omega$ $\cup \{\infty\}$.

Require $f(z)$ bounded on $\partial_\infty \Omega$, which is not satisfied by $f(z) = e^{e^z}$.

That is, require $\underbrace{\overline{\lim_{z \to a}}}_{\lim_{r \to 0} \sup \{ f(z) \mid z \in \Omega \cap B(a,r) \}} |f(z)| \le M$ for all $a \in \partial_\infty \Omega$

$$\lim_{r \to 0} \sup \{ f(z) \mid z \in \Omega \cap B(a,r) \}$$

Let's prove this latter modification for unbounded domains:

Thm: Ω: open, conn., f: analytic on Ω with $\overline{\lim_{z\to a}} |f(z)| \le M$ on $\partial_\infty \Omega$.

Then $|f(z)| \le M$ for all $z \in \Omega$.

pf: Given any $\delta > 0$. Show $\Phi := \{ z \in \Omega \mid |f(z)| > M + \delta \}$ is empty.

Since $|f(z)|$ is continuous, then Φ is open.

Further since $\overline{\lim_{z\to a}} |f(z)| \le M$ on $\partial_\infty \Omega$, then $\exists\ B(a,r)$ s.t.

$|f(z)| < M + \delta \quad \forall\ z \in \Omega \cap B(a,r)$. $\Rightarrow$ $\overline{\Phi} \subset \Omega$.

and $\overline{\Phi}$ bounded since true for nbhds of ∞ as well.

$\Rightarrow$ $\overline{\Phi}$ compact.

Apply Maximum modulus principle to Φ.

then noting $\partial \Phi$ consists of points z s.t. $|f(z)| = M + \delta$

b/c $\overline{\Phi} \subseteq \{ z \mid |f(z)| \ge M + \delta \}$, either $f(z)$ constant on Φ or Φ empty.

Even if f constant, Φ is empty. //

Since $\overline{\lim_{z\to a}} |f(z)| \le M$ on $\partial_\infty \Omega$