Cauchy's integral theorem :

$f(z)$ analytic on open disk D, γ closed curve. $a \notin \gamma$

then $$n(\gamma, a) \cdot f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a}\, dz$$

and we could extend this to f analytic on $D - \{\text{finite \# of pts } \xi_j\}$ with $\lim_{z \to \xi_j} (z - \xi_j) f(z) = 0$

so long as $a \neq \xi_j$ some j.

What happens at exceptional points?

Thm: $f(z)$ analytic on $\Omega \setminus \{a\}$. Then $f(z)$ may be extended to an analytic function on Ω (agreeing with f on $\Omega \setminus \{a\}$) $\Longleftrightarrow$ $\lim_{z \to a} (z-a) f(z) = 0$.

($\Rightarrow$) f analytic at $a \Rightarrow f$ continuous at a, so $\lim_{z \to a} f(z) = f(a)$

and hence $\lim_{z \to a} (z-a) f(z) = 0$.

Note: Any such extension is moreover unique, since we require f continuous at a, so $f(a) = \lim_{z \to a} f(z)$.

($\Leftarrow$) By Cauchy integral formula (which applies in generalized form) (since $\lim_{z \to a} (z-a) f(z) = 0$)

then $$f(z) \overset{(*)}{=} \frac{1}{2\pi i} \int_C \frac{f(\xi)\, d\xi}{\xi - z}$$ where C suff. small circle about a.

However, last time we proved that, for f continuous on γ

$$\int_\gamma \frac{f(\xi)\,d\xi}{(\xi - z)}$$ defines an analytic function for all z in "regions det'd by γ"

interiors of simple closed paths cut out by γ.

So RHS of $(*)$ makes sense at $z = a$.

If we take this to be the newly assigned value with $\gamma = C$: circle about a

$$f(a) := \int_C \frac{f(\xi)\,d\xi}{(\xi - a)}$$, the resulting function is analytic on C, agreeing with $f(z)$ on $\Omega \setminus \{a\}$. //

Use this result to do Taylor approximation:
(f analytic on Ω)

$$f(z) = f(a) + (z-a)\,f_1(z)$$ since $\frac{f(z) - f(a)}{z - a}$ satisfies conditions of theorem

for some function f_1 analytic in the region Ω containing a.

Recursively define

$$f_1(z) = f_1(a) + (z-a)\,f_2(z) \quad \ldots$$

so after times:

$$f(z) = f(a) + (z-a)\,f_1(a) + \cdots + (z-a)^n\, f_n(z) \qquad (**)$$

NOTE!

and by differentiating both sides

$$f_n(a) = \frac{f^{(n)}(a)}{n!}$$

What can we say about remainder term $f_n(z)$? Use Cauchy integral formula to estimate it.

$$f_n(z) = \frac{1}{2\pi i} \int_C \frac{f_n(\xi)\, d\xi}{\xi - z}$$ C: circle containing a. $z \in C$

Now $$f_n(\xi) = \underbrace{\frac{f(\xi)}{(\xi-a)^n}}_{\text{only this term remains in substit. into integral above.}} - \left[\frac{f(a)}{(\xi-a)^n} + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!\,(\xi-a)} \right]$$

for any of these terms $\frac{f^{(j)}(a)}{j!} \int_C \frac{1}{(\xi-a)^{n-j}(\xi-z)} = 0$ by partial fractions.

(similar to hw. problem you did last week)

$\Rightarrow$ $$f_n(z) = \frac{1}{2\pi i} \int_C \frac{f(\xi)\, d\xi}{(\xi-a)^n(\xi-z)}$$

We can use this to prove if $f^{(n)}(a) = 0 \ \forall n$, then $f = 0$: on open, conn. set Ω.

$|f(z)| \leq M$ on our circle C (compact)

$\Rightarrow$ $$|f_n(z)| \leq \frac{M}{R^{n-1}\cdot(R - |z-a|)}$$ if C has radius R.

But if first $n-1$ derivatives vanish:

$$f(z) = f_n(z)\,(z-a)^n \Rightarrow |f(z)| \leq \left(\frac{|z-a|^n}{R^n}\right) \cdot \frac{MR}{R-|z-a|}$$

since $z \in C$, $\frac{|z-a|^n}{R^n} \to 0$ as $n \to \infty$

Hence $f(z) = 0$ on C. (use topological arg to show identically 0 on Ω)

Topological argument: $\Omega = E_1 \cup E_2$, E_1: pts at which all derivatives of f vanish

E_2: pts for which some derivative is non-zero

Just shown E_1 open (vanishes on small nbhd containing a)

and E_2 open since all derivatives are continuous.

But Ω connected $\Rightarrow$ E_1 or E_2 empty. //

So if f not identically 0, if $f(a) = 0$, then some derivative $f^{(k)}(a) \neq 0$ some k. Smallest such k is called the order of the zero. Write $f(a) = (z-a)^k f_k(z)$ with $f_k(z) \neq 0$ at $z = a$.

So zeros of analytic functions are isolated.

(since $f_k(z)$ analytic $\Rightarrow$ continuous so non-zero in nbhd of a.)

In this way, analytic functions are like polynomials.