Calculating the winding #: (Do this for $a = 0$. By change of coordinates, no loss of generality)

γ : closed curve

$0 \notin \gamma$.

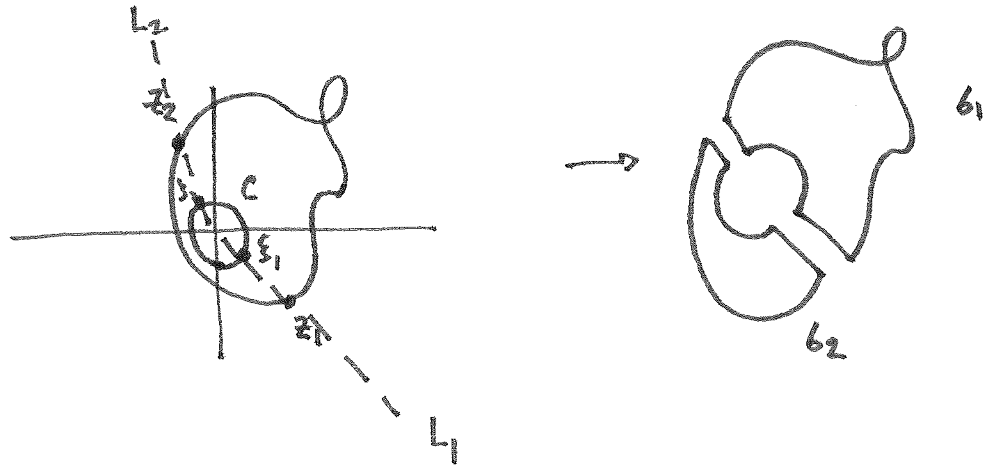
Pick $z_1, z_2 \in \gamma$ suppose $\gamma\big|_{[z_1, z_2]} =: \gamma_1$ avoids neg. real axis

$\mathrm{Im}(z_1) < 0$ $\qquad \gamma\big|_{[z_2, z_1]} =: \gamma_2$ avoids pos. real axis.

$\mathrm{Im}(z_2) > 0$

then $\quad n(\gamma, 0) = 1$.

Picture:



$C_1 = C\big|_{[L_1, L_2]}$ $\qquad$ claim: C_1 avoids neg. real axis

$C_2 = C\big|_{[L_2, L_1]}$ $\qquad$ C_2 avoids pos. real axis

But σ_1 avoids neg real axis, so 0 and ∞ in same connected comp. def'd by σ_1

$\Rightarrow n(\sigma_1, 0) = n(\sigma_1, \infty) = 0$.