Last time, we defined

$$\int_\gamma f\, dz = \int_a^b \underbrace{f(z(t))\, z'(t)}\, dt$$

then split integrand into Re, Im parts evaluate each as real Riemann integral

where $z(t)$ parametrizes the piecewise diff. arc γ.
$t \in [a,b]$

Using this definition, we can make convenient additional definitions:

$$\int_\gamma f\, \overline{dz} \overset{def}{:=} \overline{\int_\gamma \overline{f}\, dz}$$

and then further define:

$$\int_\gamma f\, dx \overset{def}{:=} \frac{1}{2}\left(\int_\gamma f\, dz + \int_\gamma f\, \overline{dz} \right) \qquad (*)$$

$$\int_\gamma f\, dy \overset{def}{:=} \frac{1}{2i}\left(\int_\gamma f\, dz - \int_\gamma f\, \overline{dz} \right)$$

We can give alternate definitions for these if we write $f = u + iv$, u,v real
$z(t) = x(t) + iy(t)$
x,y real

Work through details, then we see that

$$\int_\gamma f\, dx = \int_a^b f(z(t))\, x'(t)\, dt$$

and similarly

$$\int_\gamma f\, dy = \int_a^b f(z(t))\, y'(t)\, dt$$

which justify the names we gave to these integrals in $(*)$.

So could have started with definitions of complex line integrals

$$\int_\gamma p\,dx, \quad \int_\gamma q\,dy \quad \text{for arbitrary cx.-valued functions } p, q.$$

Thm: The line integral $\displaystyle\int_\gamma p\,dx + q\,dy$ for $\overset{\text{any}}{\gamma} \subseteq \Omega$: conn., open

depends only on endpts of γ $\iff$ $\exists\ u(x,y)$ on Ω s.t.

$$\frac{\partial u}{\partial x} = p, \quad \frac{\partial u}{\partial y} = q$$ (say $p\,dx + q\,dy$ is "exact differential")

image of function under differential d

Remark: Saying the line integral depends only on endpts $\iff$

integral over a closed curve is 0.

"$\Rightarrow$" if depends only on endpts, γ closed curve, then $\gamma, -\gamma$ have

same endpts. So $\displaystyle\int_\gamma f = \int_{-\gamma} f$. But we proved $\displaystyle\int_\gamma f = -\int_{-\gamma} f$

$$\Rightarrow \int_\gamma f = 0.$$

"$\Leftarrow$" if int. over closed curve is 0. γ_1, γ_2 two paths from $\gamma_1(a)$ to $\gamma_1(b)$.

then $\gamma_1 - \gamma_2$ is closed path $\Rightarrow \displaystyle\int_{\gamma_1 - \gamma_2} f = 0 \Rightarrow \int_{\gamma_1} f = \int_{\gamma_2} f.$

So we could have rephrased the theorem in terms of integrals over

closed paths.

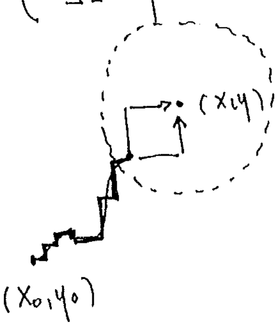
pf. of theorem: if $\exists\, u$ s.t. $\frac{\partial u}{\partial x} = p$, $\frac{\partial u}{\partial y} = q$, then

$$\int_\gamma p\,dx + q\,dy = \int_a^b \partial u/_{\partial x}(x(t),y(t))\,x'(t)\,dt + \frac{\partial u}{\partial y}(x(t),y(t))\,y'(t)\,dt$$

$$= u(x(b),y(b)) - u(x(a),y(a)).$$

To show "$\Rightarrow$": Define $u(x,y) = \int_\gamma p\,dx + q\,dy$ where γ is any rectilinear path joining (x_0,y_0) to (x,y). Function is well-defined since
fixed point in Ω

the line integral depends only on endpts. of γ. We may choose a rectilinear path whose last segment is either vertical or horizontal. (Ω: open: so can take γ:



)

If last segment of γ is horizontal: y fixed, x variable

so $u(x,y)$, as function on this segment, is

$$u(x,y) = \int_{x_0}^{x} p(x,y)\,dx + \text{const.} \Rightarrow \frac{\partial u}{\partial x} = p$$ by Fund. Thm. of Calc.

(similarly $\partial u/_{\partial y} = q$ by taking last segment vertical.) //

Corollary: $\int_\gamma f\,dz = \int_\gamma f\,dx + if\,dy$ (f: continuous) depends only on endpoints $\Longleftrightarrow \exists\, F$ s.t. $\frac{\partial F}{\partial x} = f$, $\frac{\partial F}{\partial y} = if$

(i.e. $\frac{\partial F}{\partial x} = -i\frac{\partial F}{\partial y} \Longleftrightarrow F$ analytic)

That is, $\int_\gamma f\, dz = 0$ on closed curves γ in Ω $\iff$ f is derivative of an analytic function in Ω.

End of last time: $\int_{C_R} z^n\, dz = 0$ unless $n = -1$.

C_R: circle of radius r

$\left(z^n = \frac{d}{dz} \left(\frac{1}{n+1} z^{n+1} \right) \text{ if } n \neq -1 \right)$

so in fact integral is 0 over any closed curve. Not just a circle.

Earlier, we said we'll prove

F analytic, then F has derivatives of all orders.

so in fact, $F' = f$ should be analytic as well.

Want to prove: $\int_\gamma f\, dz = 0$ on (sufficiently nice) closed curves γ in Ω $\iff$ f is analytic in Ω.

topological constraints on γ

example of $\int_{C_r} \frac{1}{z}\, dz$ shows some add't'l hypothesis required.

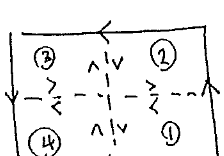
First step: R: rectangle in $\mathbb{C}$

∂R: boundary of R

f: analytic on R. (R: closed, so this means f analytic on an open set $\Omega \supseteq R$)

Cauchy's Thm on Rectangles: $\int_{\partial R} f(z)\, dz = 0$

pf:



then $\int_{\partial R} f = \int_{\partial R_1} f + \cdots + \int_{\partial R_4} f$

Suppose traversing ∂R in counterclockwise orientation.

with (at least) one of $\left| \int_{\partial R_i} f \right| \geq \frac{1}{4} \left| \int_{\partial R} f \right|$

pick smallest index i. Repeat.

Get nested sequence of rectangles $R_{i_1} \supset R_{i_2} \supset \cdots$

with $\left| \int_{\partial R_{i_n}} f \right| \geq \frac{1}{4^n} \left| \int_{\partial R} f \right|$ and $\{ R_{i_k} \} \longrightarrow z^*$: pt in rectangle as $k \to \infty$

Given $\epsilon > 0$, pick $\delta > 0$ s.t. 2 conditions are satisfied :

1. $f(z)$ analytic for all $|z - z^*| < \delta$ (since $f(z)$ analytic on $\Omega \supseteq R$ open)

2. $$\underbrace{\left| \frac{f(z) - f(z^*)}{z - z^*} - f'(z^*) \right|} < \epsilon \quad \text{if} \quad |z - z^*| < \delta .$$

(i.e. $\left| f(z) - f(z^*) - f'(z^*)(z - z^*) \right| < \epsilon \left| (z - z^*) \right|$)

For n suff. large, $R_n \subseteq \{ z \mid |z - z^*| < \delta \}$. For these n,

$$\int_{\partial R_n} 1 \cdot dz = 0 \quad (\text{since } 1 \text{ is deriv. of } z)$$

$$\int_{\partial R_n} z \cdot dz = 0 \quad (\text{since } z \text{ is deriv. of } \tfrac{1}{2} z^2)$$

} so true by previous thm.

$$\Rightarrow \int_{\partial R_n} f(z)\, dz = \int_{\partial R_n} \left[f(z) - f(z^*) - f'(z^*)(z - z^*) \right] dz$$

so $$\left| \int_{\partial R_n} f(z)\, dz \right| \leq \epsilon \cdot \int_{\partial R_n} |z - z^*| \, |dz| \leq \epsilon \cdot d_n \cdot p_n$$

Since $|z - z^*| \leq d_n$: length of diagonal of R_n, p_n : perimeter of R_n

Thus $$\left| \int f \right| \leq 4^n \cdot \left| \int f \right| \leq 4^n \cdot \epsilon \cdot d_n \cdot p_n = 4^n \cdot \epsilon \, \frac{d}{2^n} \cdot \frac{p}{2^n}$$

(ε : arbitrary) //