Ben Brubaker – Complex Analysis – 8701 FALL 2013

Text : L. Ahlfors , "Complex Analysis" , 3rd Ed. (learn from masters!)

Course Website : math.umn.edu / ~ brubaker / 8701.html

+ moodle site for grades / discussions (linked from course page. should be automatically registered ...)

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Today : Discuss some of basics of complex numbers and preview first portion of course.

(roughly, what we intend to cover before first and only midterm on ~~Friday~~ Wednesday, October ~~12~~ 16, in class.)

Missed classes:
M. Sept. 9
Week of Oct. 14

Short Answer : Get through as much of Ahlfors' book as possible.

PLEASE READ !

- Skim notes before class (~ 30 min) pages are listed on class syllabus. (linked from course page)
- Read carefully after class, write pen/paper working out details, comparing to in-class notes (2-4 hrs.) work problems.

(Exercises in Ahlfors are great ! Do as many as you can. Course notes scanned / posted on web. Can't promise they'll be exactly what happens in class, but very close)

Weekly problem sets will consist mostly of problems from Ahlfors' book. Due Fridays. First one due in 9 days : Fri. Sep. 14.

Long Answer: Several ways to define complex numbers $\mathbb{C}$

We'll take the following: ordered pairs of real #'s (a,b)

with component-wise addition and multiplication given by

$$(a_1, b_1)\cdot(a_2, b_2) = (a_1a_2 - b_1b_2,\ a_1b_2 + b_1a_2)$$

with subfield $(a,0)$, $a \in \mathbb{R}$ isomorphic to $\mathbb{R}$

check this satisfies axioms of a field (assoc., comm., distrib. add + mult. id., inverses)

Easier in many cases to define $i := (0,1)$

and express cx. numbers in form $a+bi$. Note $(0,1)^2 = (-1,0)$

so think of i as "$\sqrt{-1}$". But so is $-i$..

so gives solution to $z^2+1=0$.

(This is historic first appearance of cx. numbers – solutions to polynomial equations which were irreducible over $\mathbb{R}$.)

e.g. $x^3 - 15x - 4$. Want algorithm for finding roots, might have to allow for expressions involving $\sqrt{-1}$.

Bombelli (1560)

$$\sqrt[3]{2+\sqrt{-121}} + \sqrt[3]{2-\sqrt{-121}} = 4.$$

Other ways to define $\mathbb{C}$.

- matrices of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ (w/ usual matrix ops.)

- as field $\mathbb{R}[x]/(x^2+1)$, degree 2 Galois extension over $\mathbb{R}$

  ring of polys in one var. x with coeffs in $\mathbb{R}$

  $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \langle \bar{\ } \rangle$

  complex conjugation $a+bi \mapsto a-bi$

The early ordered pair notation is suggestive: (a,b) visualized as point in $\mathbb{R}^2$. Addition is just vector addition and size:

$$|a+bi|^2 = a^2+b^2 \quad (\text{usual distance } \cancel{\text{...}} \text{ function in } \mathbb{R}^2)$$

if $z = a+bi$, then $|a+bi|^2 = z\cdot\bar{z}$ (and so $z^{-1} = \dfrac{\bar{z}}{|z|^2}$ one of our field axioms.)

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \| \;\; |z|^2$

(i.e. $d(z,w) := |z-w|$)

Once we have distances, we can do analysis.

Take functions $f: \mathbb{C} \to \mathbb{C}$ which are differentiable:

i.e. $\displaystyle\lim_{h\to 0} \frac{f(z+h) - f(z)}{h}$ exists (at a point, or for all $z \in \mathbb{C}$ or some other open set $\Omega \subseteq \mathbb{C}$)

just as in real case. This definition is much more restrictive, as h may follow any path to z. Might think cx. differentiable functions (terminology: holomorphic / analytic) should behave in same way as real differentiable functions. Not so!

<u>Mantra</u>: Complex differentiable functions exhibit a miraculous amount of structure!

<u>Exs.</u> (1) Liouville's Thm (A. p. 121) $f: \mathbb{C} \to \mathbb{C}$ differentiable, bounded then f is constant.

(Not so for reals. Many examples: sin, cos, etc.)

(2) <u>Thm</u> (A p. 121) If f is diff., then it is infinitely differentiable. (Not so over reals. Whole hierarchy:

$$\underset{\text{continuous}}{C^0(I)} \supset \underset{\text{diff.}}{D^1(I)} \supset \underset{\substack{\text{continuous}\\ \text{first deriv}}}{C^1(I)} \supset \cdots$$

e.g. $f(x) = \int_0^x |t|\, dt \ : \mathbb{R} \to \mathbb{R}$

is in $C^1(I)$, not $D^2(I)$.

which collapses for holomorphic functions.

(3) If it is ∞-ly differentiable, does it equal its Taylor expansion on open nbhd. of point? Yes for complex diff., No for $\mathbb{R}$.

(A. p. 125) $\quad$ (Example over $\mathbb{R}$: $f(x) = \begin{cases} e^{1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$)

We'll even see why this fails by studying complex-valued generalization of f and understanding how it blows up at $x=0$ (or rather $z=0$)

(4) If $f: \mathbb{C} \to \mathbb{C}$ differentiable, and $f(\frac{1}{n}) = 0 \quad \forall\, n \in \mathbb{N}$

then $f \equiv 0$ (special case of A. p. 127)

(Not so over $\mathbb{R}$. Example : $f(x) = \begin{cases} x^2 \sin(\pi/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$)

Might think all of this means cx. analysis not so interesting - not enough functions. And yet we'll see some stunning applications.

Tool : Complex integration. (Note page numbers above. All clustered together in our book. Reason: Immediately follows Cauchy Integral Formula.)

Basic Fact : If f is holomorphic on $\Omega \subseteq \mathbb{C}$, then for "appropriate" closed paths $\gamma \in \Omega$, $\displaystyle\int_\gamma f(z)\, dz = 0$.

Applications: Already noted cx. analysis may clarify aspects of real analysis. (Probably already seen this in partial fractions method of integration, Fourier analysis.) Also appears in number theory.

Prime number theorem: $\pi(x) \sim \frac{x}{\log x}$ where

$\pi(x) := \#$ of primes $\leq x$. Here "$f \sim g$" means $\lim_{x \to \infty} \frac{f(x)}{g(x)} = 1$.

We will see important examples of real integrals in probability evaluated via cx. integration

"$\sim$" reads "is asymptotic to"

Idea: Put prime counting function into cx-valued generating function.

Study cx. function via cx. integration.

Naive guess: $\sum_{n=1}^{\infty} \pi(n) z^n$ but this doesn't converge so well ...

better: $\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^z}$ $\qquad \Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ some } k \\ 0 & \text{else.} \end{cases}$