By definition, we have

$$\cos i = \frac{e^{i\cdot i} + e^{-i\cdot i}}{2} = \frac{e^{-2} + e}{2} = \frac{1}{2e} + \frac{e}{2}.$$

Similarly, we compute

$$\sin i = \frac{e^{i\cdot i} - e^{-i\cdot i}}{2} = \frac{1}{2ie} - \frac{e}{2i} = \frac{-i}{2e} + \frac{ie}{2} = i\left(\frac{e}{2} - \frac{1}{2e}\right).$$

Finally, we compute

$$\begin{aligned}\tan(1+i) = -i\frac{e^{i(1+i)} - e^{-i(1+i)}}{e^{i(1+i)} + e^{-i(1+i)}} &= -i\frac{e^{i}e^{-1} - e^{-i}e}{e^{i}e^{-1} + e^{-i}e} \\ &= -i\frac{\frac{e^{i}}{e} - \frac{e}{e^{i}}}{\frac{e^{i}}{e} + \frac{e}{e^{i}}} \\ &= -i\frac{\frac{e^{2i}}{ee^{i}} - \frac{e^{2}}{ee^{i}}}{\frac{e^{2i}}{ee^{i}} + \frac{e^{2}}{ee^{i}}} = -i\frac{e^{2i} - e^{2}}{e^{2i} + e^{2}}.\end{aligned}$$

■

2.3.4 #5 Find the real and imaginary parts of $\exp(e^{z})$.

If $z = x + iy$, then we have

$$e^{e^{z}} = e^{e^{x+iy}} = e^{e^{x}\cos y + ie^{x}\sin y} = e^{e^{x}\cos y}\cos(e^{x}\sin y) + ie^{e^{x}\cos y}\sin(e^{x}\sin y).$$

Therefore the real part of $e^{e^{z}}$ is

$$e^{e^{x}\cos y}\cos(e^{x}\sin y)$$

and the imaginary part is

$$e^{e^{x}\cos y}\sin(e^{x}\sin y).$$

■

2.3.4 #6 Determine all values of 2^i, i^i and $(-1)^{2i}$.

Here goes more computation, where each k ranges over the integers:

$$2^i = e^{i\log 2} = e^{i(\log|2|+i\arg 2)} = e^{i\log|2|}e^{-2\pi k}.$$

$$i^i = e^{i\log i} = e^{i(\log|i|+i\arg i)} = e^{i\log|1|}e^{-\left(\frac{\pi}{2}+2\pi k\right)} = e^{-\left(\frac{\pi}{2}+2\pi k\right)}.$$

$$(-1)^{2i} = e^{2i(\log|-1|+i\arg(-1))} = e^{2i\log|1|}e^{-2(\pi+2\pi k)} = e^{2\pi+4\pi k}.$$

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■

2.3.4 #10 Show that the roots of the binomial equation $z^n = a$ are the vertices of a regular polygon.

The roots of the binomial equation are given by

$$z = \sqrt[n]{|a|}e^{i\frac{\arg a+2\pi k}{n}},$$

where $1 \leq k \leq n$ Therefore, they all lie on a circe centered at the origin with radius $\sqrt[n]{|a|}$, and are equally spaced angles since the arguments differ by $\frac{2\pi k}{n}$. That is, they form a regular n-gon. ■

3.1.2 #1 If S is a metric space with distance function $d(x,y)$, show that S with the distance function $\delta(x,y) = \frac{d(x,y)}{1+d(x,y)}$ is also a metric space.

First we see if $\delta(x,y) = 0$, then $\frac{d(x,y)}{1+d(x,y)} = 0$, and so $d(x,y) = 0$, and $x = y$.

Next, if $x = y$, then

$$\delta(x,y) = \frac{d(x,y)}{1+d(x,y)} = \frac{0}{1+0} = 0.$$

Finally, since $d(x,y)$ is a distance function, we have

$$\begin{aligned}
d(x,z) &\le d(x,y) + d(y,z) \\
d(x,z) &\le d(x,y) + d(y,z) \\
&\quad + 2d(y,z)\,d(x,y) + d(x,z)\,d(x,y)\,d(y,z) \\
d(x,z)\,d(x,y) + d(x,z)\,d(y,z) & \\
+d(x,z)\,d(x,y)\,d(y,z) + d(x,z) &\le d(x,y) + d(y,z) + 2d(y,z)\,d(x,y) \\
&\quad + d(x,z)\,d(x,y)\,d(y,z) + d(x,z)\,d(x,y) \\
&\quad + d(x,z)\,d(y,z) + d(x,z)\,d(x,y)\,d(y,z) \\
d(x,z)\left(1 + d(x,y) + d(y,z) + d(x,y)\,d(y,z)\right) &\le d(x,y)\left(1 + d(y,z) + d(x,z) + d(x,z)\right) \\
&\quad + d(y,z) \\
&\quad \left(1 + d(x,y) + d(x,z) + d(x,z)\,d(x,y)\right) \\
d(x,z)\left(1_d(x,y)\right)\left(1 + d(y,z)\right) &\le d(x,y)\left(1 + d(x,z)\right)\left(1 + d(y,z)\right) \\
&\quad + d(y,z)\left(1 + d(x,z)\right)\left(1 + d(x,y)\right) \\
\frac{d(x,z)\left(1 + d(x,y)\right)\left(1 + d(y,z)\right)}{\left(1 + d(x,z)\right)\left(1 + d(x,y)\right)\left(1 + d(y,z)\right)} &\le \frac{d(x,y)\left(1 + d(x,z)\right)\left(1 + d(y,z)\right)}{\left(1 + d(x,z)\right)\left(1 + d(x,y)\right)\left(1 + d(y,z)\right)} \\
&\quad + \frac{d(y,z)\left(1 + d(x,z)\right)\left(1 + d(x,y)\right)}{\left(1 + d(x,z)\right)\left(1 + d(x,y)\right)\left(1 + d(y,z)\right)} \\
\frac{d(x,z)}{1 + d(x,z)} &\le \frac{d(x,y)}{1 + d(x,y)} + \frac{d(y,z)}{1 + d(y,z)} \\
\delta(x,z) &\le \delta(x,y) + \delta(y,z).
\end{aligned}$$

Thus, the new distance function satisfies the triangle inequality, and S is a metric space. ■

3.1.2 #7 Show the accumulation points of any set form a closed set.

Consider a set X, its closure X^- and the set of accumulation points X^A. If we say X^I are the isolated points of X, the complement of the set of accumulation points is

$$(X^A)^C = (X^- \setminus X^O)^C$$

which is the set of points $x \notin X^-$ or $x \in X^I$. If $x \in (X^-)^C$, which is open, so x is an interior point of the complement of X^A. If x is isolated in X, then there is a neighborhood of x whose itersection with X is just x. Therefore, x is in the interior of the commplement of X^A. Thus, the complement of the set of accumulations points is open, and the set of accumulation points is closed. ∎

3.1.3 #3 Prove that the closure of a connected set is connected.

Consider a connected set X and assume there exist $A, B \subset X^-$ such that $A \cap B = \emptyset$ and $X^- = A \cup B$. Then there are disjoint sets $C = A \cap X$ and $D = B \cap X$ such that $C \cup D = X$. Since X is connected, then either C or D must be empty, so that one of their closures is empty. Therefore, X^- must be connected. ∎

3.1.3 #4 Let A be the set of points $(x, y) \in \mathbb{R}^2$ with $x = 0$, $|y| \leq 1$, and let B be the set with $x > 0$, $y = \sin\left(\frac{1}{x}\right)$. Is $A \cup B$ connected?

Since $y = \sin\left(\frac{1}{x}\right)$ is continuous when $x > 0$, it is connected. We can see that any poin in A is a limit point of B, since any neighborhood of point in A contains an infinite number of points in B. Thus, the closure of B is $A \cup B$. By the previous problem, the closure of a connected set is connected, so $A \cup B$ is connected. ∎

3.1.4 #3 Use compactness to porve that a closed bounded set of real numbers hasd a maximum.

Let X be closed and bounded in $\mathbb{R}$ so that X is compact. Assume $\sup X = x$ is not the maximum of X, and define an open cover of X by

$$U = \cup_{n \in \mathbb{N}} \left\{ X \cap \left(-\infty, x - \frac{1}{n} \right) \right\}.$$

Then for any $x_0 \in X$, there is some n_0 such that $x - \frac{1}{n_0} > x_0$, and this is in fact a cover. However, any finite subset of U will have a maximum $N \in \mathbb{N}$ so that there is some $x_0 \in X$ such that $x_0 > x - \frac{1}{N}$, and x_0 is not covered. That is, there is no finite subcover, contradicting the compactness of X. Thus, there must be a maximum of every compact set in the real numbers. ∎

3.1.4 #4 If $E_1 \supset E_2 \supset E_3 \supset \ldots$ is a decreasing sequence of nonempty compact sets, then the intersection $\cap_{n=1}^{\infty} E_n$ is not empty. Show by example that this need not be true if the sets are merely closed.

We know the arbitrary intersection of closed sets of closed, so $\cap_{n=1}^{\infty} E_n$ is closed. Futhermore, $\cap_{n=1}^{\infty} E_n \subset E_1$ since the sequence is decreasing, so the intersection is bounded, and thus compact. Now consider a sequnce $\{x_j\}$ where $x_j \in A_j$ for each n. Then there exists a convergent subcover with its limit in A. In fact, if we remove the first k sets, we will see that there is a convergent subsequence with limit in A_{k+1}. That is, the limit of the subsequence is in all A_j, and must also be in the intersection. Therefore, the intersection is nonempty. ∎

3.1.5 #1 Construct a topological mapping of the open disk $|z| < 1$ onto the whole plane.

Consider a mapping $f : \{z : |z| < 1\} \to \mathbb{C}$ given by $f(z) = \frac{z}{1-|z|}$. Note that this mapping is continuous where $|z| \neq 1$, so that it is continuous on our domain. To see the map is injective, let $f(z) = f(w)$. Then we have

$$\frac{z}{1-|z|} = \frac{w}{1-|w|}$$

which is equivalent to

$$\frac{z}{w} = \frac{1-|z|^2}{1-|w|}.$$

Since $\frac{1-|z|^2}{1-|w|^2}$ is purely real, it must be that $z = xw$ for some $x \in \mathbb{R}$. In that case,

$$\frac{z}{1-|z|^2} = \frac{xz}{1-|xz|} = \frac{xz}{1-x|z|}.$$

Rearranging, we get

$$x - x|z| = 1 - x|z|$$

which is only true if $x = 1$. That is, $w = z$, so that f is injective.

To show surjectivity, note that any ball $|z| < \epsilon < 1$ maps to a ball $|w| < \frac{1}{1-\epsilon^2}$. Thus, taking a limit as $\epsilon \to 1$, we get $|w| \to \infty$, so the function in surjective. Thus we have a bijective contuous function, and it is a homeomorphism. ∎

3.1.5 #3 Prove that every continuous one-to-one mapping of a compact space is topological.

Assume a mapping f of a compact (metric) space X is continuous and one-to-one. Then the image of f is also compact. If we consider a closed subset of X, then it must also be compact, so its image is compact. However, every compact subset of a compact set in a metric space is closed, so the image is closed. That is, f maps closed sets to closed sets, which means the inverse image of f is continuous. Futhermore, since f is one-to-one onto its image, the inverse is also one-to-one, so f is topological. ∎

3.2.2 #1 Give a precise definition of a single-valued branch of $\sqrt{1+z}+\sqrt{1-z}$ in a suitable region, and prove that it is analytic.

First begin by noting the principal branch of $\sqrt{z}$ is given by $\mathbb{C}\setminus(-\infty,0]$. Then shifting the values of z we find that $\mathbb{C}\setminus(-\infty,-1]$ is a single-valued branch of $\sqrt{1+z}$ and $\mathbb{C}\setminus[1,\infty)$ is a single valued branch of $\sqrt{1-z}$. Taking the intersection, so that both functions are single-valued, a single valued branch cut for $\sqrt{1+z}+\sqrt{1-z}$ is $\mathbb{C}\setminus\{(-\infty,-1],[1,\infty)\}$.

restrict arg(z) (-1)

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The derivative is given by $\frac{1}{\sqrt{1+z}}-\frac{1}{\sqrt{1-z}}$. Then the derivative is undefined at $z=\pm 1$, but those are excluded from our domain. In addition, we know $\sqrt{z}$ is analytic, and sums of analytic functions are again analytic, so $\sqrt{z}$ is given by $\mathbb{C}\setminus(-\infty,0]$ is analytic. ■

3.2.2 #2 Give a precise definition of a single-valued branch of $\log(\log z)$ in a suitable region, and prove that it is analytic.

The principal branch of $\log z$ is given by $\mathbb{C}\setminus(-\infty,0]$. Its image is the slit plane $\{z=x+iy:|y|<\pi\}$. Removing the interval $(-\infty,0]$ from the image would give another single-valued function, and the inverse image of $(-\infty,0]$ under $\log z$ is $(0,1]$. Therefore, a single-valued branch of $\log(\log z)$ is $\mathbb{C}\setminus(-\infty,1]$.

Its derivative is $\frac{1}{z\log z}$, which is undefined at $z=0,1$, which are not in our domain. Furthermore, $\log z$ is analytic and the composition of analytic functions is analytic, so $\log(\log z)$ is analytic. ■

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