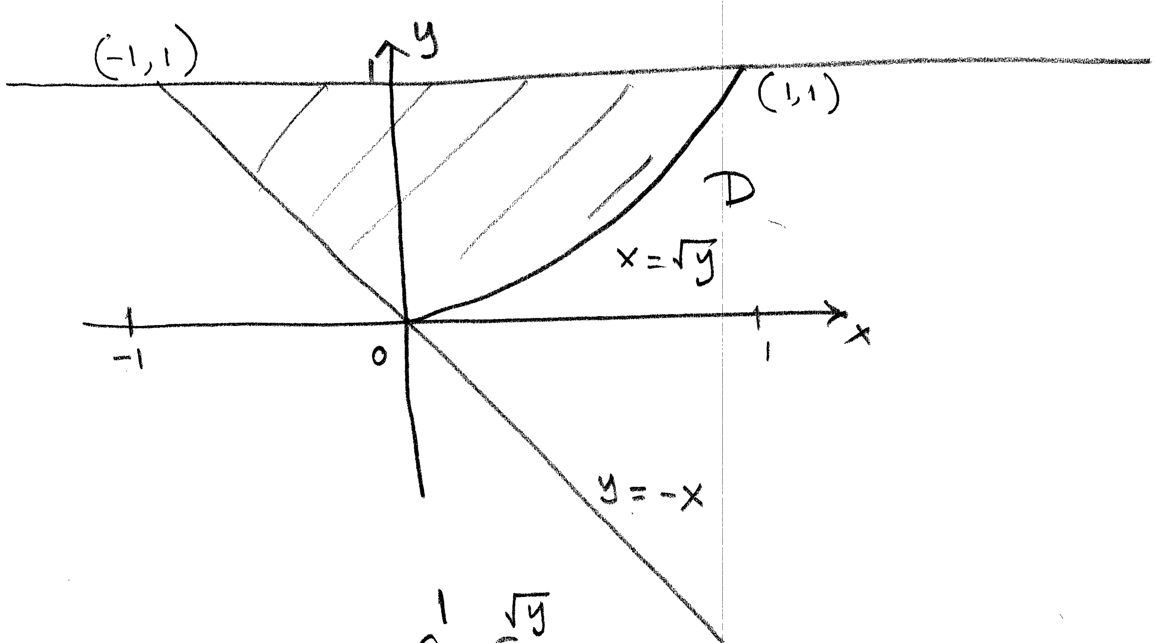
Answers to Practice Problems for Midterm 2

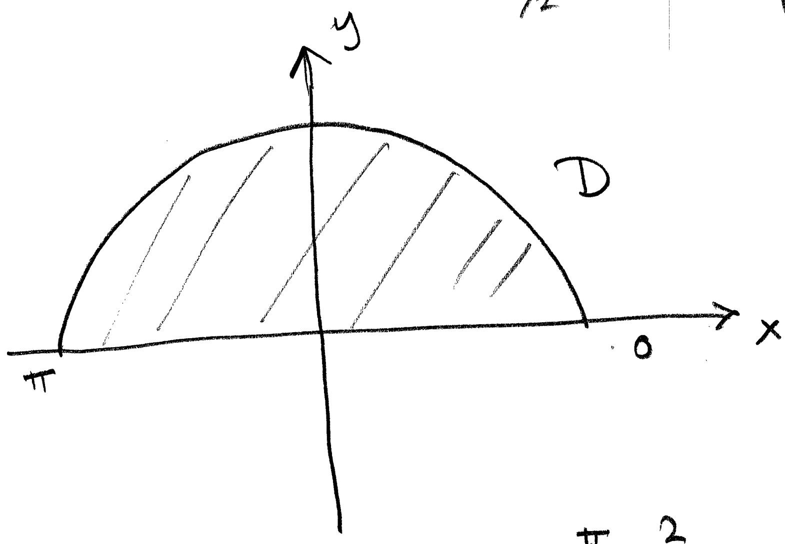


$$\text{Area}(D) = \int_0^1 \int_{-y}^{\sqrt{y}} 1 \, dx\, dy$$

$$= \int_0^1 \left(y^{1/2} + y\right) dy$$

$$= \frac{y^{3/2}}{3/2} + \frac{y^2}{2} \Big|_0^1 = \frac{2}{3} + \frac{1}{2}.$$

$$= \frac{7}{6}.$$



$$D = \{(x,y) \mid y \geq 0, \; x^2+y^2 \leq 4\}$$

$$= \{(r,\theta) \mid 0 \leq \theta \leq \pi, \; 0 \leq r \leq 2\}$$

$$\iint_D e^{-\frac{1}{2}(x^2+y^2)} \, dA = \int_0^{\pi} \int_0^2 e^{-\frac{1}{2}r^2} r \, dr \, d\theta$$

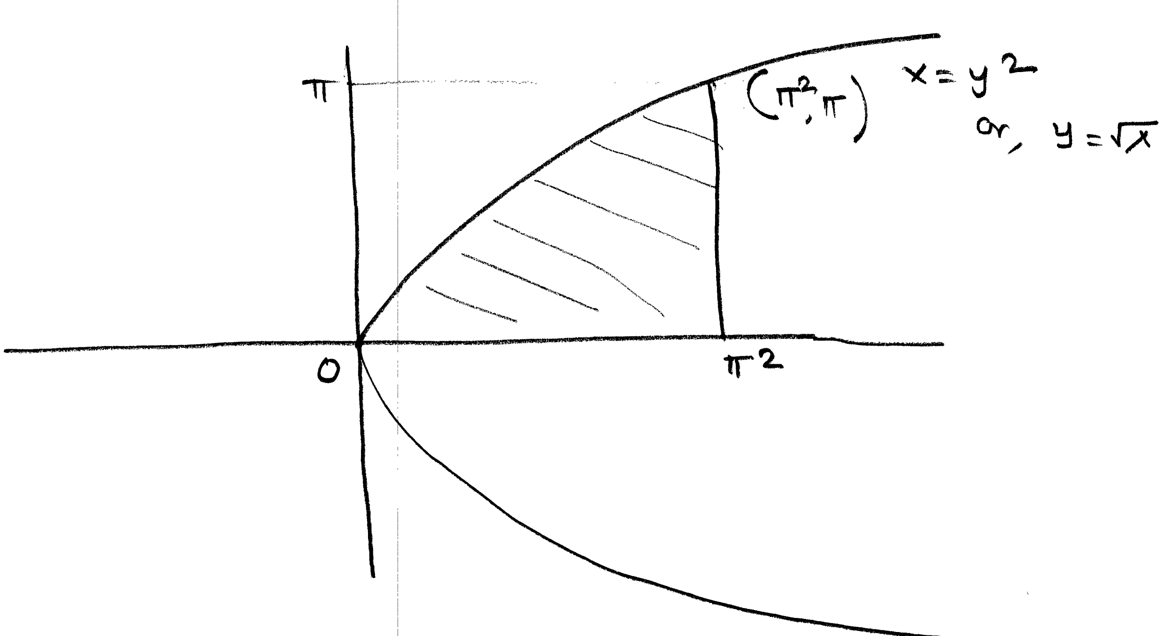
$$= \int_0^{\pi} d\theta \times \int_0^2 e^{-\frac{1}{2}r^2} r \, dr = \pi \times \int_0^2 e^{-u} \, du$$

$r^2/2 = u \quad r\,dr = du$

$= \pi \; x - e^{-u} \Big|_0^2 \quad = \pi \left(1 - e^{-2}\right).$

$I = \int_0^{\pi} \int_{y^2}^{\pi^2} y \cos(x^2) \, dx \, dy$

$D = \left\{ (x, y) \;\middle|\; 0 \le y \le \pi, \quad y^2 \le x \le \pi^2 \right\}$



$I = \int_{x=0}^{\pi^2} \int_{y=0}^{\sqrt{x}} y \cos(x^2) \, dy \, dx \quad = \int_0^{\pi^2} \cos(x^2) \cdot \frac{y^2}{2} \Big|_0^{\sqrt{x}} dx$

$= \frac{1}{2} \int_0^{\pi^2} x \cos(x^2) \, dx$

$x^2 = u$

$2x \, dx = du$

$= \frac{1}{4} \int_0^{\pi^4} \cos(u) \, . \, du \quad = \frac{1}{4} \times \sin u \Big|_0^{\pi^4}$

$= \frac{1}{4} \left(\sin(\pi^4) - \sin(0) \right) =$

$= \boxed{\frac{1}{4} \sin(\pi^4) \, .}$

4. $f(x,y) = 2x^3 + xy^2 + 5x^2 + y^2 + 2$

$f_x = 6x^2 + y^2 + 10x$

$f_y = 2xy + 2y$

$f_x = 0$, $f_y = 0$

$2xy + 2y = 0 \Rightarrow y(1+x) = 0$

So, either $y = 0$ or $x = -1$.

Case I $y = 0$: $f_x = 6x^2 + y^2 + 10x = 0$

$\Rightarrow 6x^2 + 10x = 0$

or, $x = -5/3$ or 0.

Case II :- $x = -1$: then $f_x = y^2 - 4 = 0$

or, $y = \pm 2$

So, the critical points:

$(x,y) = (0,0)$, $(-5/3, 0)$, $(-1, 2)$ & $(-1,-2)$.

$f_{xx} = 12x + 10$, $f_{xy} = 2y$, $f_{yy} = 2x + 2$

$D = f_{xx} f_{yy} - f_{xy}^2$

at $(x,y) = (0,0)$: $f_{xx} = 10$, $f_{xy} = 0$, $f_{yy} = 2$

So, $f_{xx} = 10 > 0$ $D = 20 > 0$

So, $(0,0)$ is a local minimum point.

at $(x,y) = (-5/3, 0)$: $f_{xx} = -20$, $f_{xy} = 0$, $f_{yy} = -\frac{4}{3}$.

$D = \frac{80}{3} > 0$ $\quad f_{xx} < 0$

$(-5/3, 0)$ is a local maximum.

at $(x,y) = (-1, 2)$: $f_{xx} = -2$, $f_{xy} = 4$, $f_{yy} = 0$

$D = -16 < 0$: saddle point

at $(x,y) = (-1,-2)$ $\quad f_{xx} = -2$, $f_{xy} = -4$, $f_{yy} = 0$

$D = -16 < 0$: saddle point.

5. distance from any point (x, y, z) from the origin $= \sqrt{x^2+y^2+z^2}$.

Want to minimize d or equivalently $d^2 = x^2+y^2+z^2$ subject to the constraint $y^2 = 1 + xz$

$f(x,y,z) = x^2+y^2+z^2$ subject to $g(x,y,z) = xz+1-y^2 = 0$ — (*)

$\nabla f = \lambda \nabla g$.

$2x = \lambda z$ — (1)

$2y = \lambda.(-2y)$ — (2)

$2z = \lambda . x$ — (3)

From (2) :- $y(1+\lambda) = 0$

So, either $\lambda = -1$ or $y = 0$.

Case I: $\lambda = -1$

(1) $\Rightarrow$ $2x = -z$ — (4)

(3) $\Rightarrow$ $2z = -x$ — (5)

$\Bigg\} \Rightarrow -4z = -z$ or, $z = 0$ & $x = 0$.

By (*), $y^2 = 1$ or, $y = \pm 1$.

$(0, \pm 1, 0)$

Case II :- $y = 0$

① × ③ $\Rightarrow$ $4xz = \lambda^2 xz$

$\Rightarrow$ either $xz = 0$ or, $\lambda = \pm 2$

If $xz = 0$ & $y = 0$ then $xz + 1 - y^2 = 1$

So, $xz \neq 0$ & hence $\lambda = \pm 2$.

① $\Rightarrow$ $x = \pm z$

If $x = -z$ from (*), $-x^2 + 1 = 0$ or $x = \pm 1$

$x = 1,\ z = -1$

$x = -1,\ z = +1$

If $x = z$ from (*) $x^2 + 1 = 0$ no solution

$(1, 0, -1)$ $(-1, 0, 1)$

$f(0, \pm 1, 0) = 1$

$f(1, 0, -1) = f(-1, 0, 1) = 2$

The closest points are $(0, \pm 1, 0)$.

6. (a) $f(x,y) = xy$ $\qquad g(x,y) = x^2 + 2y^2 - 1 = 0$

Lagrange : $\nabla f = \lambda \nabla g$

$$y = \lambda \cdot 2x \quad \text{——} \ (1)$$

$$x = \lambda \cdot 4y \quad \text{——} \ (2)$$

$$x^2 + 2y^2 = 1 \quad \text{——} \ (3)$$

(1) gives $y = 2\lambda x$. Plug in (2),

$$x = 4\lambda \cdot 2\lambda x \quad \Rightarrow \quad x(1 - 8\lambda^2) = 0.$$

So, either $x = 0$ or $\lambda = \pm 1/\sqrt{8}$

Case I : $x = 0$. From (2) $y = 0$.

hence $x^2 + 2y^2 = 0 \neq 1$.

So, $x \neq 0$.

Case II. $\lambda = \pm 1/\sqrt{8}$

From (2), $x = \pm\sqrt{2}\, y$ —— (4)

From (3), $(\pm\sqrt{2}\, y)^2 + 2y^2 = 1$

or, $4y^2 = 1$

$y = \pm 1/2 \quad \Rightarrow \quad x = \pm \frac{1}{\sqrt{2}}$

$(x,y) = \left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{2}\right)$. four points

$$\left.\begin{array}{l} f\left(\frac{1}{\sqrt{2}}, \frac{1}{2}\right) = f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{2}\right) = \frac{1}{2\sqrt{2}} \\ f\left(\frac{1}{\sqrt{2}}, -\frac{1}{2}\right) = f\left(-\frac{1}{\sqrt{2}}, +\frac{1}{2}\right) = -\frac{1}{2\sqrt{2}} \end{array}\right\} \quad **$$

$\left(\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$, $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{2}\right)$: abs max

$\left(\frac{1}{\sqrt{2}}, -\frac{1}{2}\right)$, $\left(-\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$: abs min

on $x^2 + 2y^2 = 1$

(b). critical point f inside D.

$\nabla f = 0$ $\quad y = 0$, $x = 0$

$(x, y) = (0, 0)$.

$f(0,0) = 0$

Comparing with $**$,

$\left(\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$, $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{2}\right)$; abs max

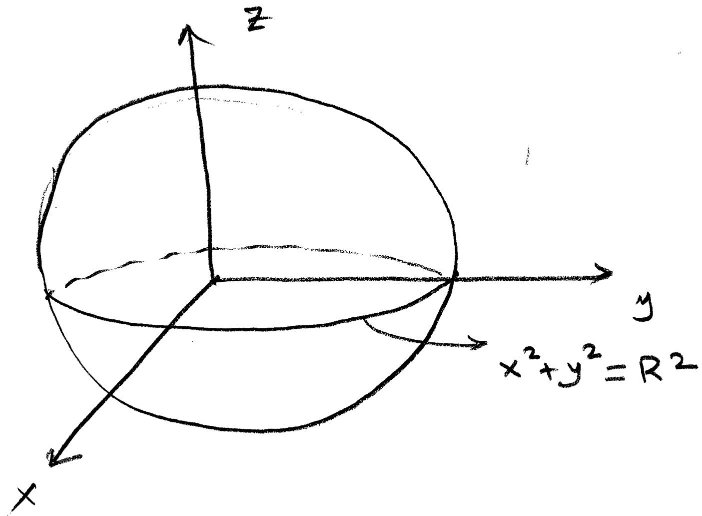
$\left(\frac{1}{\sqrt{2}}, -\frac{1}{2}\right)$, $\left(-\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$: abs min

on D.

7.(a) $E = \{(x,y,z) : x^2+y^2+z^2 \le R^2\}$

$= \{(r,\theta,z) : 0 \le \theta \le 2\pi,\ 0 \le r \le R,$

$-\sqrt{R^2-r^2} \le z \le \sqrt{R^2-r^2}\}$



$z^2 \le R^2 - (x^2+y^2)$

or, $z^2 \le R^2 - r^2$

or, $-\sqrt{R^2-r^2} \le z \le +\sqrt{R^2-r^2}$

$$\mathrm{Vol}(E) = \int_0^{2\pi}\int_0^R\int_{-\sqrt{R^2-r^2}}^{+\sqrt{R^2-r^2}} 1 \cdot dz\ r\,dr\,d\theta$$

$$= 2\pi \times \int_0^R 2\sqrt{R^2-r^2} \cdot r \cdot dr$$

$R^2 - r^2 = u$

$-2r\,dr = du$

$$= 2\pi \int_{R^2}^{0} -\sqrt{u}\ du$$

$$= 2\pi \int_0^{R^2} u^{1/2} \cdot du = 2\pi \left.\frac{u^{3/2}}{3/2}\right|_0^{R^2}$$

$$= \frac{4\pi}{3} R^3.$$

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$$E = \{ (\rho, \theta, \phi) : 0 \le \rho \le$$

$$(E) = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} 1. \ \rho^2 \sin\phi$$

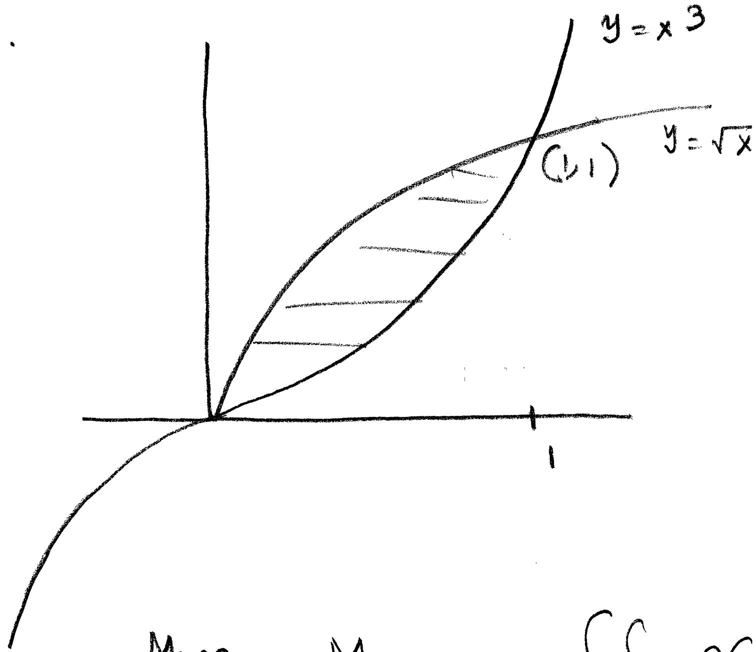
$$= \int_0^{2\pi} 1.d\theta \quad \times \int_0^{\pi} \sin\phi \, .d\phi$$

$$= 2\pi \quad \times \quad -\cos\phi \Big|_0^{\pi}$$

$$= 2\pi \quad \times 2 \quad \times \frac{R^3}{3}$$

(11)

8.



D

Mass $\quad M = \iint_D \rho(x,y)\, dA$

$= \int_0^1 \int_{x^3}^{\sqrt{x}} x^2 y \cdot dy \cdot dx$

$= \int_0^1 x^2 \cdot \frac{y^2}{2}\Big|_{x^3}^{\sqrt{x}} \cdot dx \quad = \frac{1}{2}\int_0^1 x^2 \cdot (x - x^6) \cdot dx$

$= \frac{1}{2}\int_0^1 (x^3 - x^8) dx \quad = \frac{1}{2}\left(\frac{1}{4} - \frac{1}{9}\right) \quad = \frac{5}{2 \times 36}$

$= \frac{5}{72}$.

y- Moment $\quad M_y = \iint_D y\, \rho(x,y)\, dA$

$= \int_0^1 \int_{x^3}^{\sqrt{x}} x^2 y^2 \cdot dy\, dx$

$= \int_0^1 x^2 \cdot \frac{y^3}{3}\Big|_{x^3}^{\sqrt{x}} \cdot dx \quad = \frac{1}{3}\int_0^1 x^2 (x^{3/2} - x^9)\, dx$

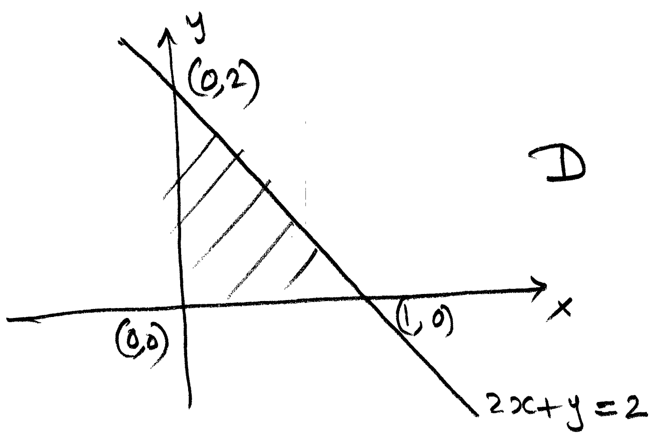
$= \frac{1}{3}\int_0^1 (x^{7/2} - x^{11})\, dx$

$= \frac{1}{3}\left(\frac{2}{9} - \frac{1}{12}\right) = \frac{1}{3} \times \frac{5}{36} \quad = \frac{5}{108}$

y-Comp. of the center of mass

$$\frac{M_y}{M} = \frac{\frac{5}{108}}{\frac{5}{72}} = \frac{72}{108} = \frac{2}{3}.$$

9. The plane $2x + y + z = 2$ intersects at the line $2x + y = 2$.



$$z = 2 - 2x - y = f(x, y)$$

$$\text{Surface area} = \iint_D \sqrt{1 + f_x^2 + f_y^2}.$$

$$f_x = -2 \quad , \quad f_y = -1$$

$$= \iint_D \sqrt{1 + (-2)^2 + (-1)^2} \; dA$$

$$= \sqrt{6} \iint_D 1.dA = \sqrt{6} \text{ Area}(D)$$

$$= \sqrt{6} \cdot \tfrac{1}{2} \times 1 \times 2$$

$$= \sqrt{6}.$$

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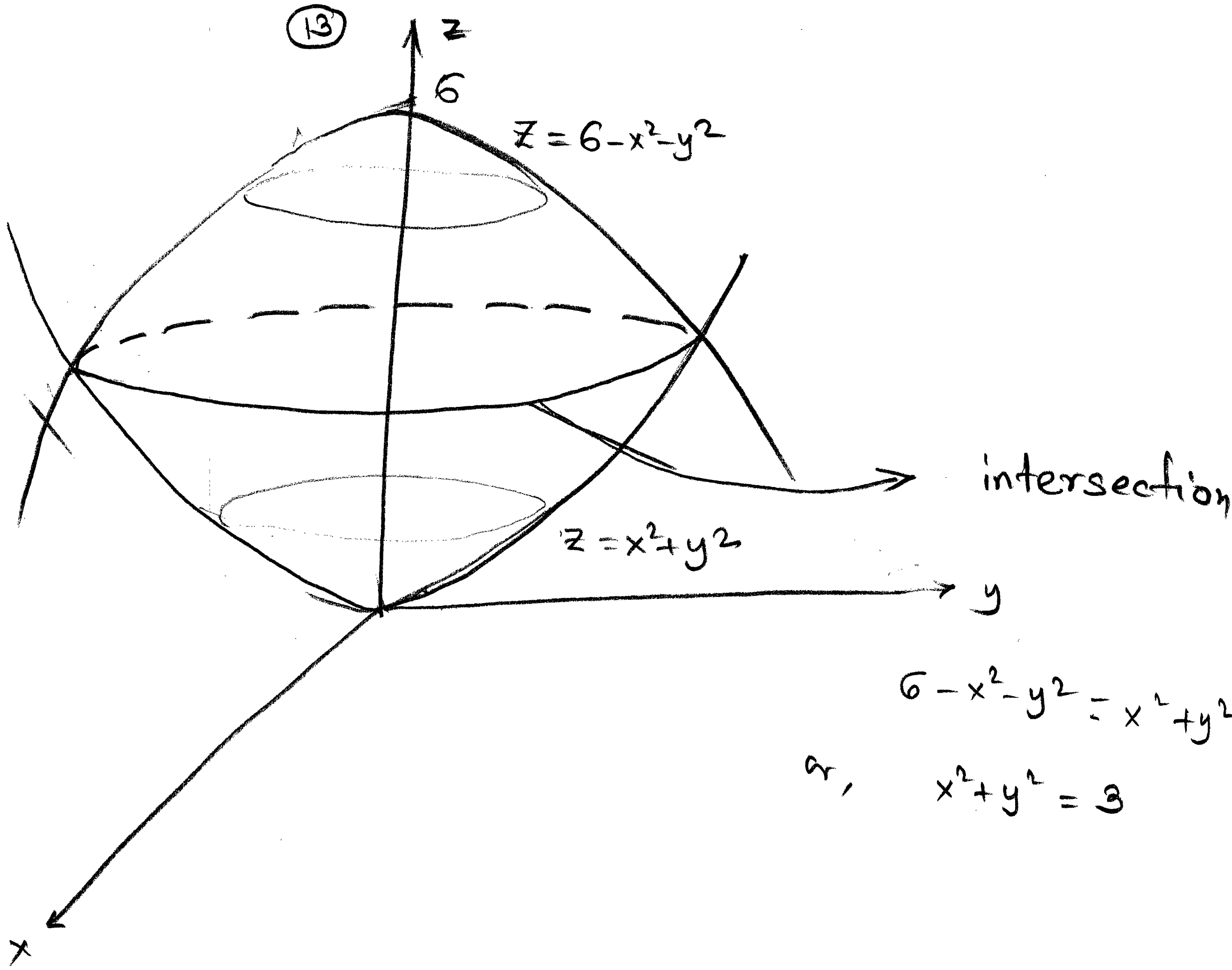
10.



$6-x^2-y^2 = x^2+y^2$

or, $x^2+y^2 = 3$

projection of E in the xy plane

D : $x^2+y^2 \le 3$

Use cylindrical coordinates, $r^2 \le 3$, $r \le \sqrt{3}$

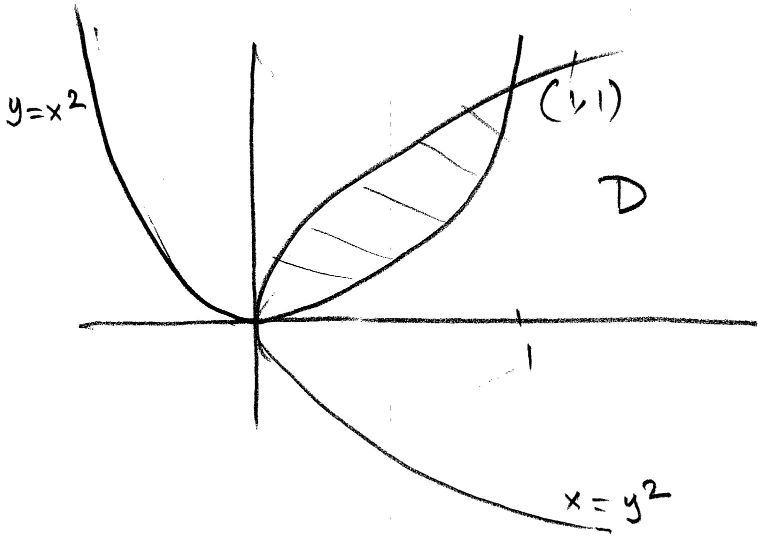
$$\text{volume} = \int_0^{2\pi}\int_0^{\sqrt{3}}\int_{z=r^2}^{z=6-r^2} 1.\,dz\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\sqrt{3}} (6-2r^2)\,r.\,dr.\,d\theta = 2\pi \times \left. \frac{6r^2}{2} - \frac{2.r^4}{4} \right|_0^{\sqrt{3}}$$

$$= 2\pi \times \left(9 - \frac{9}{2}\right) = 2\pi \times \frac{9}{2}$$

$$= 9\pi.$$

(14)



$E = \{$

$(x,y) \in D, \ 0$

$$\text{Volume} = \iint_{(x,y)\in D} \int_0^{x+y} 1 \, dz \, dx\, dy$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} (x+y) \, dy\, dx$$

$$= \int_0^1 xy + \frac{y^2}{2} \Big|_{x^2}^{\sqrt{x}} dx$$

$$= \int_0^1 x(\sqrt{x} - x^2) + \tfrac{1}{2}(x - x^4) \, dx$$

$$= \int_0^1 x^{3/2} - x^3 + \tfrac{1}{2}x - \tfrac{1}{2}x^4 \, dx$$

$$= \frac{2}{5} - \frac{1}{4} + \frac{1}{4} - \frac{1}{10} = \frac{3}{10}.$$

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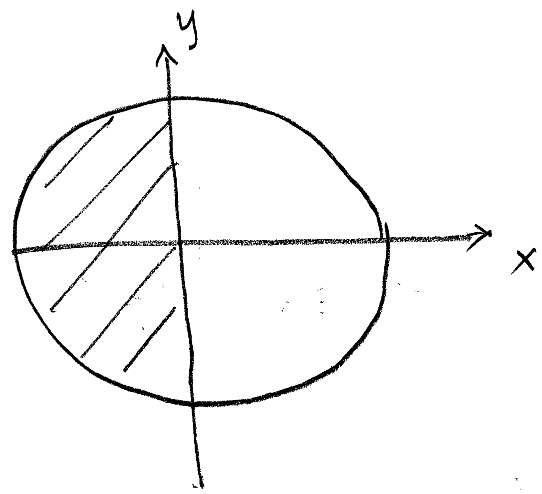
12. $$\int_{-1}^{0}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{0}^{\sqrt{1-x^2-y^2}} \frac{1}{x^2+y^2+z^2}\,dz\,dy\,dx$$

$-1 \le x \le 0$, $-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$

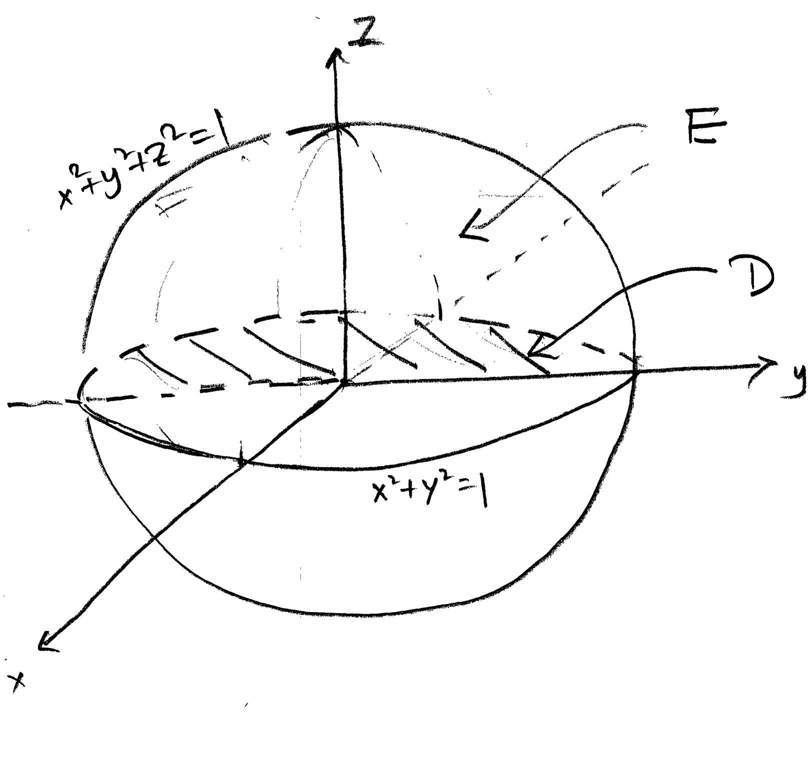
i.e. $x^2+y^2 \le 1$

D



$0 \le z \le \sqrt{1-x^2-y^2}$ $\Rightarrow$ $x^2+y^2+z^2 \le 1$

& $z \ge 0$



$z \ge 0$

$x \le 0$

Use spherical coordinates :-

$$E = \left\{ (\rho, \theta, \phi) : 0 < \rho \le 1, \quad \pi/2 \le \theta \le \quad 0 \le \phi \le \pi/2 \right\}$$

Integral =

$$\int_{\theta=\pi/2}^{3\pi/2} \int_{\phi=0}^{\pi/2} \int_{\rho=0}^{1} \frac{1}{\rho^2} \cdot \rho^2 \cdot \sin\phi \cdot d\rho \, d\phi \, . d\theta$$

$$= \int_{\pi/2}^{3\pi/2} d\theta \times \int_0^{\pi/2} \sin\phi \times \int_0^1 1 \cdot d\rho$$

$$= \pi \times -\cos\phi \Big|_0^{\pi/2} \times 1$$

$$= \pi .$$