Solutions to Sample Problems 1

1.(a) The vectors $\overrightarrow{PQ} = \langle -1, -2, -1 \rangle$ and $\overrightarrow{PR} = \langle 1, 1, 3 \rangle$ lie on the plane.

A normal vector to the plane:

$$\vec{V} = \overrightarrow{PQ} \times \overrightarrow{PR} = \langle -5, 2, 1 \rangle.$$

The plane goes through $P = (1, 0, -3)$.

The eq.n for the plane :- $-5(x-1) + 2(y-0) + 1.(z+3) = 0$

or, $5x - 2y - z - 8 = 0$.

(b). Area of $\Delta PQR = \frac{1}{2} \left| \overrightarrow{PQ} \times \overrightarrow{PR} \right|$

$$= \frac{1}{2} \times \sqrt{30}$$

2. Two points on the line (and hence on the plane)

$t = 0 \; : Q = (0, -2, 1)$ $\quad t = 1 : R = (2, -1, 0)$

$P = (1, 0, -1)$ is also on the plane.

$\overrightarrow{PQ} = \langle -1, -2, 2 \rangle$, $\quad \overrightarrow{PR} = \langle 1, -1, 1 \rangle$

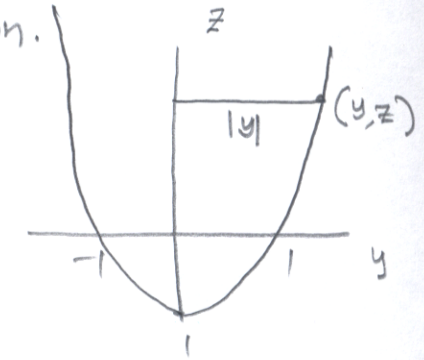
normal vector : $\overrightarrow{PQ} \times \overrightarrow{PR} = \langle 0, 3, 3 \rangle$.

The eq.n for plane: $(x-1) \cdot 0 + (y-0) \cdot 3 + (z+1) \cdot 3$

or, $y + z + 1 = 0$.

3.(a)

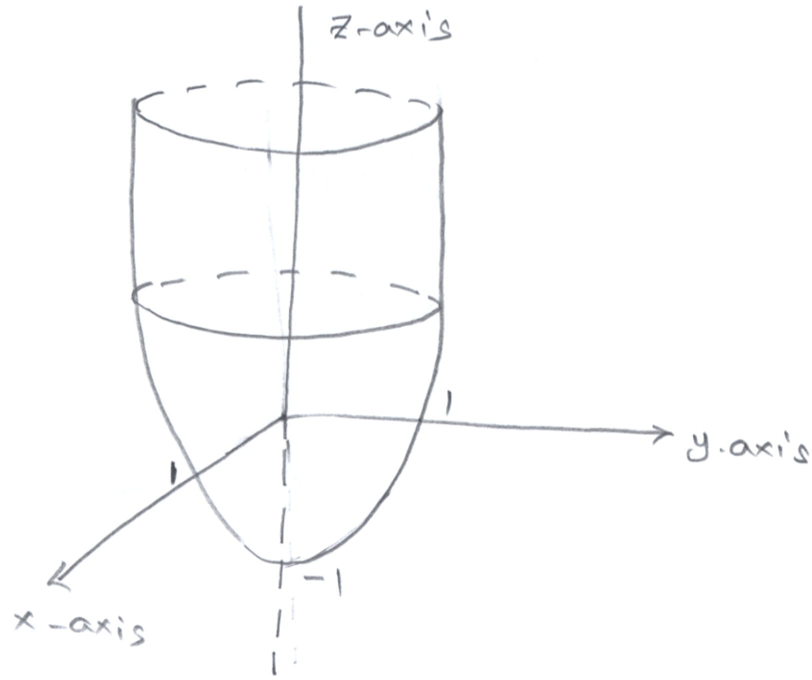
Take a point (x, y, z) on the surface. The distance from z-axis is $\sqrt{x^2+y^2}$, which replaces $|y|$ in the given equation.



yz - plane

So, the eqn of the surface:

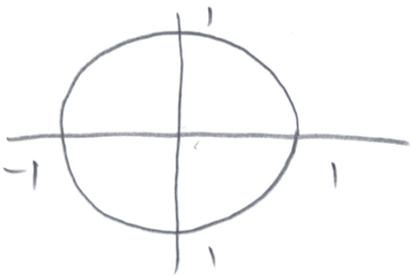
$$z = x^2+y^2-1.$$



elliptic paraboloid.

(b). Set $z=0$ in $z = x^2+y^2-1$

circle of radius 1 : $x^2+y^2=1$



4. $$\lim_{(x,y)\to(1,0)} \frac{xy-y}{(x-1)^2+2y^2}$$ ③

$$= \lim_{(\bar{x},y)\to(0,0)} \frac{\bar{x}\,y}{\bar{x}^2+2y^2}$$

Set $\bar{x} = x-1$.

$x \to 1$, $\bar{x} \to 0$

We take limit along $y = m\bar{x}$

$$\lim_{\bar{x}\to 0} \frac{m\bar{x}^2}{\bar{x}^2+2m^2\bar{x}^2} = \lim_{\bar{x}\to 0} \frac{m}{1+2m^2} = \frac{m}{1+2m^2}$$

$$= \begin{cases} 0 & m=0 \\ 1/3 & m=1 \end{cases}$$

The limit does not exist.

5. $f(x,y) = (x^2-y^2)e^{xy}$

$f_x = 2x\,e^{xy} + (x^2-y^2)\,y\,e^{xy}$

$f_y = -2y\,e^{xy} + (x^2-y^2)\,x\,e^{xy}$

$f_{xx} = 2e^{xy} + 2xy\,e^{xy} + 2xy\,e^{xy} + (x^2-y^2)y^2 e^{xy}$

$f_{xy} = 2x^2 e^{xy} + (x^2-3y^2)\,e^{xy} + (x^2-y^2)xy\,e^{xy}$

$f_{yy} = -2e^{xy} - 2xye^{xy} + x^2(x^2-y^2)\,e^{xy}$
$\quad - 2y.x\,e^{xy}$

$f_{xx}(-1,1) = 2e^{-1} - 2e^{-1} - 2e^{-1} + 0 = -2e^{-1}$

$f_{xy}(-1,1) = 2e^{-1} - 2e^{-1} = 0$

$f_{yy}(-1,1) = -2e^{-1} + 2e^{-1} + 0 + 2e^{-1} = 2e^{-1}$.

6.

direction vector $\overrightarrow{AB} = \langle -1, 0, -1 \rangle$

parametric eqn.:

$x = 1 - t, \quad y = 2, \quad z = 3 - t.$

point of intersection with sphere:

Solve for t s.t.

$$(1-t)^2 + 2^2 + (3-t)^2 = 8$$

or, $2t^2 - 8t + 6 = 0$ or, $t^2 - 4t + 3 = 0$

or, $(t-1)(t-3) = 0$ or $t = 1$ or 3.

So, the intersection pts are:

$(0,2,2)$ and $(-2,2,0)$.

7.

$f(x,y) = \ln(2x + 3y)$.

$f_x(x,y) = \dfrac{2}{2x+3y}, \quad f_y(x,y) = \dfrac{3}{2x+3y}$

$f_x(2,-1) = 2, \quad f_y(2,-1) = 3$

tangent plane :- $\quad z - 0 = 2(x-2) + 3(y-(-1))$

or, $\quad 2x + 3y - z - 1 = 0.$

8. (a) $f(x,y) = x^4 + xy - 2y^2$

$f_x = 4x^3 + y, \qquad f_y = x - 4y$

(b) Linear approximation of f at $(1,2)$.

$$L(x,y) = f(1,2) + f_x(1,2).(x-1) + f_y(1,2)(y-2)$$

$$= -5 + 6 \times (x-1) + (-7)(y-2)$$

Estimate

$$= L(1{\cdot}01, 1{\cdot}98) = -5 + 6 \times {\cdot}01 - 7 \times (-0{\cdot}02)$$

$$= -5 + {\cdot}20 = -4{\cdot}8$$

9. $f(x,y,z) = (x+z^2)\sin(xy)$

$f_x = \sin(xy) + (x+z^2)\,y\cos(xy)$

$f_y = x(x+z^2)\cos(xy)$

$f_z = 2z\sin(xy)$

$\nabla f(1,\pi,-2) = \langle f_x(1,\pi,-2), f_y(1,\pi,-2), f_z(1,\pi,-2) \rangle$

$= \langle -5\pi, -5, 0 \rangle$

(b). The direction $= -\nabla f(1, \pi, -2)$

$= \langle 5\pi, 5, 0 \rangle$

(c). unit vector $\vec{v}$ along $\vec{u}$

: $\frac{\vec{u}}{|\vec{u}|} = \langle \frac{2}{3}, \frac{1}{3}, -\frac{2}{3} \rangle$.

$D_{\vec{v}} f(1, \pi, -2) = \nabla f(1, \pi, -2) \cdot \vec{v}$

$= -5\pi \cdot \frac{2}{3} + -5 \times \frac{1}{3} + 0 \times (-2/3)$

$= \frac{-10\pi - 5}{3} = \frac{-5(1+2\pi)}{3}$.

10. Let $P = (x, y, z)$ be an arbitary point on S.

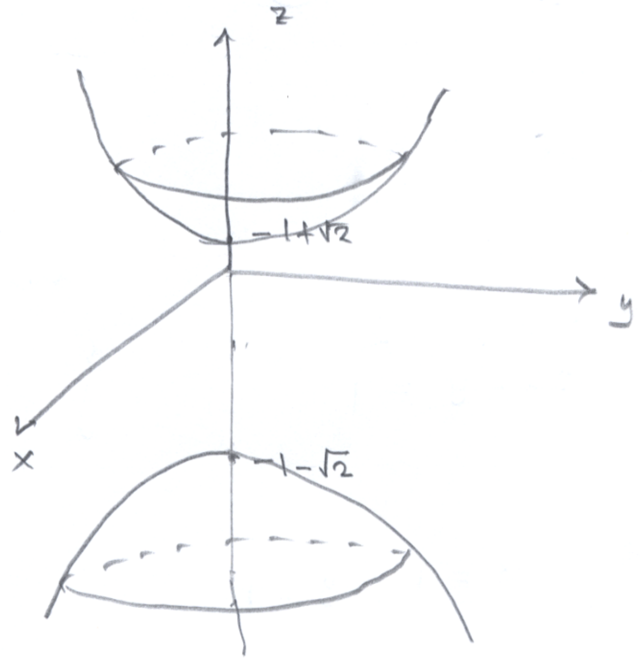
Then the distance from P to $(0,0,1)$ is

$\sqrt{x^2+y^2+(z-1)^2}$ and to xy plane is $|z|$.

$\sqrt{x^2+y^2+(z-1)^2} = \sqrt{2}\,|z|$.

Simplify: $x^2 + y^2 - (z+1)^2 = -2$ (hyperboloid of two sheets)

(7)



$$-\frac{x^2}{2} - \frac{y^2}{2} + \frac{(z+1)^2}{2} = 1$$

11. Rate of change of temperature:

$$\frac{d}{dt} T(x(t), y(t)) \Big|_{t=3}.$$

$x(t) = \sqrt{1+t}$
$y(t) = t^2 - 9$

By chain rule,

$$\frac{d}{dt} T(x(t), y(t)) = T_x(x(t), y(t)) \cdot x'(t) + T_y(x(t), y(t))\, y'(t)$$

at $t=3$

$$= T_x(2,0) \cdot \frac{1}{4} + T_y(2,0) \cdot 6$$

$$= -\frac{1}{4} + 12 = \frac{47}{4}.$$

$x'(t) = \frac{1}{2\sqrt{1+t}}$

$y'(t) = 2t$

12. $L_1: x + y - z = 2$

$L_2: 2x - y + 3z = 1$

Set $z = 0$

$$\left.\begin{aligned} x + y &= 2 \\ 2x - y &= 1 \end{aligned}\right\}$$ line of intersections of L_1 with $z=0$ and L_2 with $z=0$

Solve $x = 1,\ y = 1$

The point of intersection of L_1 and L_2 $(1, 1, 0)$.

direction vector $= \langle 1, 1, -1\rangle \times \langle 2, -1, 3\rangle$

$= \langle 2, -5, -3\rangle$

The eqn for the line:-

$$\frac{x-1}{2} = \frac{y-1}{-5} = \frac{z-0}{-3}$$